

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

Series: Universidad Complutense de Madrid

Advisers: A. Dou and M. de Guzmán

481

Miguel de Guzmán

Differentiation of Integrals in \mathbb{R}^n



Springer-Verlag
Berlin · Heidelberg · New York 1975

Author

Prof. Miguel de Guzmán
Facultad de Matemáticas
Universidad Complutense de Madrid
Madrid 3/Spain

Library of Congress Cataloging in Publication Data

Guzmán, Miguel de, 1936-

Differentiation of integrals in R^n

(Lecture notes in mathematics ; 481)

Bibliography: p.

Includes index.

1. Integrals, Generalized. 2. Measure theory.

I. Title. II. Series: Lecture notes in mathematics (Berlin) ; 481.

QA3.L28 no. 481 tQA312; 510'.8s t515'.43; 75-25635

AMS Subject Classifications (1970): 26A24, 28A15

ISBN 3-540-07399-X Springer-Verlag Berlin · Heidelberg · New York

ISBN 0-387-07399-X Springer-Verlag New York · Heidelberg · Berlin

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically those of translation, reprinting, re-use of illustrations, broadcasting, reproduction by photocopying machine or similar means, and storage in data banks.

Under § 54 of the German Copyright Law where copies are made for other than private use, a fee is payable to the publisher, the amount of the fee to be determined by agreement with the publisher.

© by Springer-Verlag Berlin · Heidelberg 1975

Printed in Germany

Offsetdruck: Julius Beltz, Hemsbach/Bergstr.

DEDICATED TO

MAYTE

Miguel₂ and Mayte₂

PREFACE

The work presented here deals with the local aspect of the differentiation theory of integrals. This theory takes its origin in the wellknown theorem of Lebesgue [1910]: Let f be a real function in $L^1(\mathbb{R}^n)$. Then, for almost every $x \in \mathbb{R}^n$ we have, for every sequence of open Euclidean balls $B(x, r_k)$ centered at x such that $r_k \rightarrow 0$,

$$\lim (1/|B(x, r_k)|) \int_{B(x, r_k)} f(y) dy = f(x) \quad \text{as } k \rightarrow \infty.$$

One could think that the fact that one takes here the limit of the means over Euclidean balls instead of taking them over other type of sets contracting to the point x might well be irrelevant. It was not until about 1927 that H. Bohr exhibited an example, first published by Carathéodory [1927], showing that intervals in \mathbb{R}^2 (i.e. rectangles with sides parallel to the axes) behave much worse than cubic intervals or circles with regard to a covering property (Vitali's lemma) that was fundamental for the result of Lebesgue. So it became a challenging problem to find out whether the replacement of Euclidean balls by intervals centered at the point x in the Lebesgue theorem would lead to a true statement or not. The first result in this direction was the so-called strong density theorem, first proved by Saks [1933], stating that if the function f is the characteristic function of a measurable set, then Euclidean balls can be replaced by intervals. Later on Zygmund [1934] showed that this can also be done if f is in any space $L^p(\mathbb{R}^n)$, with $1 < p \leq \infty$, and a year later Jessen, Marcinkiewicz and Zygmund [1935] proved that the same is valid if f is in $L(1+\log^+ L)^{n-1}(\mathbb{R}^n)$. On the other hand Saks [1934] proved that there exists a function g in $L(\mathbb{R}^n)$ such that the Lebesgue statement is false for g if one take intervals instead of balls. The *Fundamenta Mathematicae* of those years, which still remains one of the main sources of information for the theory of differentiation bears testimony to the interest of many outstanding mathematicians for this subject.

One of the important products of such activity was the surprising result that, if in the Lebesgue theorem one tries to replace circles by rectangles centered at the point x then the statement is not any more true in general even if f is assumed

to be the characteristic function of a measurable set. This was first observed by Zygmund as a byproduct of the construction by Nikodym [1927] of a certain paradoxical set.

Such findings prompted others to try to consider more general situations and to give some characterization of those families of sets that, like the Euclidean balls or the intervals, would permit a differentiation theorem similar to that of Lebesgue. The first attempts in this direction were the fundamental paper of Busemann and Feller [1934], giving such a characterization by means of a certain "halo" condition, and the paper by de Possel [1936], offering one in terms of a covering property.

In this way there arose the theory of differentiation, which as we shall have occasion to show, still presents many challenging open problems and has very interesting connections with other branches of analysis. In the present work I have tried to focus on some of the more fundamental aspects of the differentiation theory of integrals in \mathbb{R}^n . In this context the theory can be presented very concretely and with a minimal amount of terminology. Many interesting open problems, whose solution will probably lead to a better understanding of basic structures in analysis, can be stated in a way simple enough to be immediately understood by those who just know what is a Lebesgue measurable function defined on \mathbb{R}^2 .

The differentiation theory we shall present here appears as an interaction between covering properties of families of sets in \mathbb{R}^n , differentiation properties similar to that of the Lebesgue theorem, and estimations for an adequate extension of the wellknown maximal operator of Hardy and Littlewood. The whole book is a commentary on these three main subjects.

Chapter I is devoted to the main covering theorems that are used in the subject. Chapter II introduces the notions of a differentiation basis and of the maximal operator associated to it, and offers certain basic methods in order to obtain several useful estimations for this operator. Chapter III shows how closely related are the properties of the maximal operator and the differentiation properties of a basis. Chapters IV, V and VI explore some properties of several examples of differen-

tiation bases, the basis of intervals, that of rectangles, and of some special sets (convex sets and unbounded star-shaped sets). Chapter VII is devoted to the possibility of obtaining covering properties starting from differentiation properties of a basis. Finally Chapter VIII contains some considerations about a particular problem in which the author has been interested.

Each chapter is divided in sections. I have tried to offer in the main body of each section just the relevant result that has been the source of inspiration for many other further developments. In the remarks at the end of each section I give information, often rather detailed, about some extensions of the theory, without trying at all to be exhaustive. In the theory we present there are still many open problems. I have stated some of them, almost always in the remarks at the end of each section. A list of them is given at the end. Some of these problems might be easy to solve, but some others seem to be rather difficult and will perhaps require fresh ideas and new techniques in our field. I hope that some of the readers will be stimulated by such problems and so the theory will be enriched with their effort. I would certainly be very grateful for any light on these problems I might receive from them. I am very happy to say that after the first version of these notes was written, in December 1.974, some of the problems proposed in them have been solved and some others have been substantially illuminated. In the appendices at the end of this work, written by A. Córdoba, R. Fefferman and R. Moriyón one can see some of the progress that has been made. I wish to thank them for having permitted me to include in these notes their results, that will be of great value for those interested in the field. Also very recently C. Hayes has solved in a very general setting the problem proposed in page 165.

I wish to thank, first of all, Prof. Antoni Zygmund for the encouragement I have received from him to write this work and for many helpful discussions on the subject. The assistance and helpful criticism of my colleagues at the University of Madrid has been invaluable. I owe particular gratitude to C. Aparicio, M.T. Carrillo, J. López, M.T. Menárguez, R. Moriyón, I. Peral, B. Rubio and M. Wallias for many

VIII

stimulating hours we have spent discussing the topics treated here. I also wish to thank A.M. Bruckner, C. Hayes and G. V. Welland for having read the first version of these notes and for their very helpful suggestions. Paloma Rodríguez, Isi Vázquez and Pablo Mz. Alirangües were in charge of typing and preparing these notes for publication. I thank them very much for their fine job.

Miguel de Guzmán

June 1.975

Facultad de Matemáticas

Universidad Complutense de Madrid

Madrid 3, Spain

CONTENTS

CHAPTER I

SOME COVERING THEOREMS

	<u>Page</u>
1. Covering theorems of the Besicovitch type	2
2. Covering theorems of the Whitney type	9
3. Covering theorems of the Vitali type	19

CHAPTER II

THE HARDY-LITTLEWOOD MAXIMAL OPERATOR

1. Weak type (1,1) of the maximal operator	36
2. Differentiation bases and the maximal operator associated to them ..	42
3. The maximal operator associated to a product of differentiation bases	44
4. The rotation method in the study of the maximal operator	51
5. A converse inequality for the maximal operator	56
6. The space $L(1 + \log^+ L)$. Integrability properties of the maximal oper- ator	60

CHAPTER III

THE MAXIMAL OPERATOR AND THE DIFFERENTIATION PROPERTIES OF A BASIS

1. Density bases. Theorems of Busemann-Feller	66
2. Individual differentiation properties	77
3. Differentiation properties for classes of functions	81

CHAPTER IV

THE INTERVAL BASIS \mathcal{B}_2

1. The interval basis \mathcal{B}_2 does not satisfy the Vitali property	92
--	----

	<u>Page</u>
2. Saks' rarity theorem. A problem of Zygmund	96
3. A theorem of Besicovitch on the possible values of the upper and lower derivatives	100

CHAPTER V

THE BASIS OF RECTANGLES \mathcal{B}_3

1. The Perron tree. The Kakeya problem	109
2. The basis \mathcal{B}_3 is not a density basis	115
3. The Nikodym set. Some open problems	120

CHAPTER VI

SOME SPECIAL DIFFERENTIATION BASES

1. An example of Hayes. A density basis \mathcal{B} in R^1 and a function g in each L^p , $1 \leq p < \infty$, such that \mathcal{B} does not differentiate $\int g$	134
2. Bases of convex sets	137
3. Bases of unbounded sets and star-shaped sets	141
4. A problem	147

CHAPTER VII

DIFFERENTIATION AND COVERING PROPERTIES

1. The theorem of de Possel	148
2. An individual covering theorem	153
3. A covering theorem for a class of functions	158
4. A problem related to the interval basis	165
5. An example of Hayes. A basis \mathcal{B} differentiating L^q but no L^{q_1} with $q_1 < q$	166

CHAPTER VIII

ON THE HALO PROBLEM

	<u>Page</u>
1. Some properties of the halo function	179
2. A result of Hayes	180
3. An application of the extrapolation method of Yano	183
4. Some remarks on the halo problem	187

APPENDIX I

On the Vitali covering properties of a differentiation basis

by Antonio Córdoba 190

APPENDIX II

A geometric proof of the strong maximal theorem

by Antonio Córdoba and Robert Fefferman 196

APPENDIX III

Equivalence between the regularity property and the
differentiation of L^1 for a homothety invariant basis

by Roberto Moriyón 206

APPENDIX IV

On the derivation properties of a class of bases

by Roberto Moriyón 211

BIBLIOGRAPHY	215
A LIST OF SUGGESTED PROBLEMS	223
INDEX	224

SOME NOTATION

For a point x in R^n , $|x|$ means the Euclidean norm of x , i.e. if $x = (x_1, x_2, \dots, x_n)$, then

$$|x| = \left(\sum_{i=1}^n x_i^2 \right)^{1/2}.$$

For a set A in R^n , $|A|_e$ means the exterior Lebesgue measure of A , $\delta(A)$ the (Euclidean) diameter of A , ∂A the boundary of A . If A is measurable, $|A|$ denotes its measure, and A' denotes the complement of A .

For a sequence $\{A_k\}$ of subsets of R^n and a point $x \in R^n$, $A_k \rightarrow x$ (" A_k contracts to x ") means that $x \in A_k$ for each k and $\delta(A_k) \rightarrow 0$.

For a sequence $\{r_k\}$ of real numbers and $a \in R$, $r_k \uparrow a$ ($r_k \downarrow a$) means that r_k converges increasingly (decreasingly) to a .

For a family \mathcal{A} of sets of R^n , $\bigcup \{A : A \in \mathcal{A}\}$ means the set of points of R^n belonging to some set of the family \mathcal{A} .