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Glenn Schober

Univalent Functions -
Selected Topics



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PREFACE

These notes are from lectures given by the author in 1973-74 at the University of Maryland, during its special year in complex analysis. They are an attempt to bring together some basic ideas, some new results, and some old results from a new point of view, in the theory of univalent functions.

There are really two points of view that are used and intertwined in these notes. The first is to utilize a linear space framework to study sets of univalent functions as they are situated in a space of analytic functions. For example, in Chapter 7 we are interested in compactness of families of univalent functions that lie in the intersection of two hyperplanes, and in Chapter 8 we are interested in their geometry in the sense of convexity theory.

In the same spirit, we consider in Chapter 2 many of the special families of univalent functions and determine the extreme points of their closed convex hulls. This point of view seems to simplify and unify the study of their properties, for example, in solving linear extremal problems. In keeping with this point of view we give in Chapter 1 a derivation of the Herglotz representation based on Choquet's theorem. In this case the route is less elementary, but it serves to establish our point of view.

The second point of view is to study extremal problems using variational considerations. In the absence of a structural formula for a class of functions, variational methods are a very powerful tool. In Appendix C we include the boundary variation from the fundamental work of M. Schiffer, and in Chapters 10 and 11 we apply

it to solve some accessible problems and to give geometric properties of solutions to others.

Variational considerations can also be used to study quasiconformal mappings. Our treatment of quasiconformal mappings is not thorough. However, in Chapter 13 we give a very general variational procedure for families of quasiconformal mappings. Some applications are given in Chapter 14. At present, this appears to be a rapidly developing area.

A number of other topics that seem to fit in are included, e.g., the affirmative solution of the Pólya-Schoenberg conjecture, representation of continuous linear functionals, Faber polynomials, properties of quasiconformal mappings, and quasiconformal extensions of univalent functions.

One final comment about the structure of these notes: There are two kinds of problems in the text, those called exercises and those called problems. There is a distinction; namely, the author knows how to solve only the exercises. This comment was delayed until now in the hope that the reader might overlook it, and might go on to solve some of the problems.

Finally, the author wishes to acknowledge the work of Mr. Tom Whitehurst, who proofread these pages, and of Miss Julie Palmer and Mrs. Karen Barker, who typed them.

Bloomington, Indiana

Glenn Schober

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CONTENTS

CHAPTER 1.	Functions with positive real part	1
CHAPTER 2.	Special classes: convex, starlike, real, typically real, close-to-convex, bounded boundary rotation	6
CHAPTER 3.	The Pólya-Schoenberg conjecture	27
CHAPTER 4.	Representation of continuous linear functionals	34
CHAPTER 5.	Faber polynomials	39
CHAPTER 6.	Extremal length and equicontinuity	48
CHAPTER 7.	Compact families $\mathfrak{F}(D, \ell_1, \ell_2, P, Q)$ of univalent functions normalized by two linear functionals	57
CHAPTER 8.	Properties of extreme points for some compact families $\mathfrak{F}(D, \ell_1, \ell_2, P, Q)$	65
CHAPTER 9.	Elementary variational methods	79
CHAPTER 10.	Application of Schiffer's boundary variation to linear problems	92
CHAPTER 11.	Application to some nonlinear problems	112
CHAPTER 12.	Some properties of quasiconformal mappings	128
CHAPTER 13.	A variational method for q.c. mappings	138
CHAPTER 14.	Application to families of conformal and q.c. mappings	147
CHAPTER 15.	Sufficient conditions for q.c. extensions	168
APPENDIX A.	Some convexity theory	172
APPENDIX B.	Coefficient and distortion theorems	176
APPENDIX C.	Schiffer's boundary variation and fundamental lemma	181
REFERENCES		191
INDEX		199