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Continuous Convergence on C(X)

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INTRODUCTION

Some parts of functional analysis and general topology are devoted to the relationship between a completely regular topological space X and $C_{co}(X)$. By $C_{co}(X)$, we mean the $\mathbb{R}$-algebra $C(X)$ of all continuous, real-valued functions of X, equipped with the topology of uniform convergence on compact subsets of X. Many of these investigations of this relationship are hindered by the fact that the evaluation map

$$\omega : C_{co}(X) \times X \longrightarrow \mathbb{R}$$

(sending each pair $(f,p) \in C(X) \times X$ into $f(p)$) is not continuous. Another handicap is that, in general, $C_{co}(X)$ is not complete. In these notes we replace the concept of uniform convergence on compact subsets of X by the concept of continuous convergence. This type of convergence on $C(X)$ does not arise from a topology. However, it is generated by a so-called convergence structure, a notion which generalizes that of a topology. The convergence structure of continuous convergence (the continuous convergence structure) is finer than the topology of compact convergence and coincides with it when X is a locally compact topological space. The algebra $C(X)$, endowed with the continuous convergence structure, yields a complete convergence algebra, denoted by $C_c(X)$, and a continuous evaluation map

$$\omega : C_c(X) \times X \longrightarrow \mathbb{R}.$$

The convergence algebra $C_c(X)$ carries the coarsest among all the convergence structures Λ on $C(X)$ for which $\omega : C_\Lambda(X) \times X \longrightarrow \mathbb{R}$ is continuous. This fact provides $C_c(X)$ with many convenient properties. However, difficulties occur in many approximation problems especially in the context of a Stone-Weierstrass type of theorem.

The purpose of these notes is to present the foundations of the interactions between a general convergence space X and $C_c(X)$.
In Chapter o, we introduce the theory of convergence spaces (convergence spaces, continuous convergence structure, etc.). In Chapter 1, we collect some properties of $C(X)$ needed in the subsequent chapters (e.g. Stone-Čech- and realcompactifications etc.)
We demonstrate in Chapter 2, that, for a completely regular topological space X, there is (in general) no $\mathbb{R}$-vector space topology T on $C(X)$ for which

$$\omega : C_T(X) \times X \longrightarrow \mathbb{R}$$

is continuous.
In the third chapter, we exhibit and study a special class of convergence spaces, the class of c-embedded spaces. For any two such spaces X and Y, the convergence algebras $C_c(X)$ and $C_c(Y)$ are bicontinuously isomorphic iff X and Y are homeomorphic. This class turns out to be very large: Any $C_c(Z)$, where Z is an arbitrary convergence space, can be represented by $C_c(Z')$, where Z' is a c-embedded space. Among other topological spaces any completely regular topological space is c-embedded. The structure of c-embedded spaces is expressed by Schroder's theorem, which asserts that any c-embedded space is the projective limit of inductive limits of compact topological spaces. To find these topological results, we have to develop the functional analytic apparatus of $C_c(X)$. Along the way, we observe that the relationship between a c-embedded space X and $C_c(X)$ is an extension of the classical correspondence between a compact topological space Y and the Banach algebra $C_c(Y)$.
The problem of studying which convergence $\mathbb{R}$-algebras are of the form $C_c(Y)$ is the principal intent of Chapter 4. However, we restrict ourselves mostly to subalgebras of $C_c(X)$. These investigations have, of

course, some similarities to the Gelfand theory. We conclude the chapter with a study of the c-embeddedness of general function spaces. There the reader may notice that the category of c-embedded spaces is cartesian closed.

Chapter 5 is devoted to parts of the dictionary of topological properties of c-embedded spaces X and functional analytic properties of $C_c(X)$. In this context we will in particular characterize normal, separable metric and Lindelöf spaces. Necessary and sufficient conditions are given on a completely regular topological space X in order that $C_c(X)$ be representable as an inductive limit of topological $\mathbb{R}$-vector spaces.

In the appendix, we develop some results on the linear and Pontryagin duality of $C_c(X)$ and of general topological $\mathbb{R}$-vector spaces. Here the respective dual spaces will be endowed with the continuous convergence structure.

Finally we remark that the items in the Bibliography, not quoted in the text, contain interesting suplements to the material presented in these notes.