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Matts R. Essén

The cos $\pi\lambda$ Theorem

With a paper by Christer Borell

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The first eight sections of these notes are based on the second half of a course given at the University of Minnesota during the fall quarter of 1972 and at the University of Kentucky during the spring semester of 1973. During the first half of the course, an introduction to the theory of subharmonic functions was given, based on Chapters 4 and 5 in Heins (the name of an author refers to the corresponding book in the list of references). The purpose of the second half was to give a survey of recent results of Phragmén-Lindelöf type on the growth at infinity of functions subharmonic in $\underline{R}^2$. Many of these results can be extended to $\underline{R}^d$, $d \geq 3$, as is clear from the references.

The starting-point for the work reported here is the $\cos\pi\lambda$-theorem of Kjellberg [3]. For the history of the subject up to 1948, see Kjellberg [1]. The proofs given in these notes are sometimes different from those of the original papers. I have throughout used the approach of Hellsten-Kjellberg-Norstad [1] and of Essén ([2b], Lemma 3.1) (for further details, cf. sections 1 and 2). Apart from being able to use the same technique in the proofs of several known results, we are also able to extend certain $\cos\pi\lambda$-results to the situation considered by A. Baernstein [2]: the results thus obtained are new (cf. section 8).

Sections 1-4 contain results which have previously appeared elsewhere. In sections 3 and 4, I have avoided certain technical difficulties by only discussing $\underline{R}^2$. Sections 5-8 are new. Section 8 is joint work of the author and J. Lewis.

The essential tools are the Riesz representation formula for subharmonic functions and some properties of convolution inequalities. For those interested in the theory in $\underline{R}^d$, $d \geq 3$, I would like to mention that eigenfunction expansions of the Green or the Neumann function in a cone are also used.

I want to thank the Departments of Mathematics at the University of Minnesota and the University of Kentucky for the opportunity to give these lectures. I am also grateful to the chairman at the second university, Professor R. H. Cox for making it possible to get (a preliminary version of) these notes typed in Lexington.

Lexington, May 8, 1973

When I was working on the first eight sections of these notes during the spring of 1973, I received a preprint of the paper [2] of A. Baernstein. I found his extension of the $\cos\pi\lambda$-theorem very interesting and included part of his work in section 2: it also inspired J. Lewis and myself to the work reported in section 8. At the conference on Classical Function Theory in Canterbury in July 1973, A. Baernstein gave further applications of his rearrangement theorem for subharmonic functions. This time, he obtained new results on univalent functions and harmonic measures. At a series of seminars at the Mittag-Leffler Institute during the fall of 1973, I gave a survey of the work of Baernstein as well as an example of how the Baernstein technique could be extended to $\underline{R}^d$, $d \geq 3$. Most of this material is given in section 9.

At these seminars, it was noticed by Dr. Christer Borell that a result of Baernstein on estimates of harmonic measures could be extended to $\underline{R}^d$. This paper is also included in these lecture notes.

I am grateful to Albert Baernstein for letting me have preprints of his important papers in this area of research. My thanks are also due to Mrs. Beverly Mullins in Lexington and Mrs. Anna-Maria Johansson in Stockholm for their excellent typing. Finally, I would like to mention my wife Agneta without whose support and patience this work would not have been possible.

Stockholm, February 20, 1974

Matts Essén

In sections 1-3, proofs of results are given only in the case $0 < \lambda < 1/2$. The theorems are valid when $0 < \lambda < 1$. Similarly in section 4, we discuss only the case $\lambda_0 \geq 1/2$ in spite of the fact that there are results of this type also when $0 < \lambda_0 < 1/2$. Still, the proofs given here contain all the essential ideas which are needed in a complete discussion. It is not necessary for the understanding of these lecture notes to check that this statement is correct. However, it is easy for anyone who has read these notes and who wishes to know more to find the missing details in the original papers.

Let me also mention that so far, generalizations of the results in sections 1-3 to higher dimensions exist only in the case analogous to $0 < \lambda < 1/2$.

Notation. $\underline{R}$ is the real axis, $\underline{C}$ is the complex plane.

$$\Delta(a,r) = \{z \in \underline{C} : |z - a| < r\} \quad ,$$
$$C(a,r) = \{z \in \underline{C} : |z - a| = r\} \quad .$$

If u is subharmonic in a region $\Omega \subset \underline{C}$, let

$$M(r,u) = \sup u(z), \quad |z| = r, \quad z \in \Omega \quad ,$$
$$m(r,u) = \inf u(z), \quad |z| = r, \quad z \in \Omega \quad .$$

If it is clear from the context what subharmonic function we consider, we sometimes write $M(r)$ instead of $M(r,u)$. This remark applies also to other functionals than $M(r)$ and $m(r)$. Let $\partial\Omega$ be the boundary of Ω. If $\zeta \in \partial\Omega$ and u is subharmonic in Ω, define

$$u(\zeta) = \limsup u(z), \; z \to \zeta, \quad z \in \Omega \; .$$

If the region Ω is unbounded, we also introduce the order $\rho_0 = \limsup_{r \to \infty} \log M(r,u)/\log r$ and the lower order $\lambda_0 = \liminf_{r \to \infty} \log M(r,u)/\log r$ of u.

Contents