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Groups of Cohomological Dimension One

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INTRODUCTION

Free groups have cohomological dimension one, so it is natural to ask whether the converse holds. This question became of extra interest after it was shown that the similar result holds for pro-p groups.

In 1968 Stallings [17] showed that finitely generated groups of cohomological dimension one are free, and in 1969 Swan [19], using Stallings' work, solved the general problem.

THEOREM A A group of cohomological dimension one (over some ring with unit) is free (provided it is torsion-free).

Stallings and Swan also proved another theorem with an analogue for pro-p groups.

THEOREM B A torsion-free group containing a free subgroup of finite index is free.

This follows immediately from Theorem A and the following result of Serre.

THEOREM C Let R be a commutative ring with unity, G a group with a subgroup H of finite index. If G has no R-torsion (e.g. if G is torsion-free) then G and H have the same cohomological dimension over R.

These notes, based on lectures given at King's College, London, give a completely self-contained account of these theorems. An elementary knowledge of combinatorial group theory and homological algebra is needed, but the theorems of Kuroš and Gruško on free products are proved.

The notes differ from the papers of Stallings and Swan in several significant details, among them the following:

i) the theory of ends is given in the algebraic form due to the author [2] ;

ii) a key lemma for Stallings' structure theorem for groups with infinitely many ends is proved by Dunwoody's method [3];

iii) this structure theorem is given the proof recently obtained by a research student at Queen Mary College;

iv) some of Swan's homological arguments are replaced by more explicit discussion of the augmentation ideal I_G of a group G ;

v) Theorem A is relativised to give a result implying the following theorem;

<u>THEOREM D</u> <u>Let</u> H <u>be a subgroup of a free group</u> G. <u>Then</u> H <u>is a free factor of</u> G <u>iff</u> $I_H G$ <u>is a summand of</u> I_G.

My thanks are due to C. R. Leedham-Green for his careful reading of these notes; in particular, for providing me with an additional supply of commas.

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