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Stratified Polyhedra

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Introduction

The present paper is a revised version of a much longer work (which had fewer results) entitled "Block Bundle Sheaves". This is an obsolete phrase for "stratified polyhedra"; and since my work has been referred to under the old title, I mention it as a sort of subtitle of the present work. During the process of revision, I profited greatly from lecturing in a graduate seminar at M.I.T., since it was necessary to describe technical definitions and procedures intuitively. I talked about my difficulties to many friends, especially Ralph Reid, my sister Ellen and Dennis Sullivan. Above all, I am indebted to Colin Rourke, who suggested the present method of defining stratifications, and who urged me to change terminology to conform better with Thom's theory [28] of "ensembles stratifiés". I am also glad to express my gratitude to William Browder.

The two facts about block bundles, as defined by Rourke and Sanderson, which make them so important are:

1. if M is a p.ℓ. submanifold of Q, then M has a normal block bundle in Q, which is in some sense unique;

2. there is a relative transversality theorem: if M, N are submanifolds of Q, then there is an arbitrarily small isotopy f_t of Q such that $f_1 N$ is block transverse to M; moreover if bdy N is already block transverse to bdy M in bdy Q, then we may require f_t to be the identity on bdy Q.

That M has a block bundle neighbourhood in Q was shown by a simple construction: roughly speaking, take a simplicial triangulation B of Q in which M is covered by a full subcomplex A. Let B' be a first derived subdivision of B. For each simplex $s \in A$, let

$$s^*B = \hat{s} * \ell k(s, B') \quad \text{(the simplicial join);}$$
$$s^*A = \hat{s} * \ell k(s, A')$$

where $\hat{s} \in B'$ is the barycentre of s. Then s^*B is a block over s^*A, and $\{s^*B : s \in A\}$ forms the required block bundle.

This construction can equally well be performed when M and Q are no longer manifolds but general polyhedra. It seemed reasonably to hope that by following the proofs of Rourke and Sanderson one could prove

1. some sort of uniqueness result, and
2. a relative block transversality theorem.

It is the main purpose of the present paper to carry out this program.

In Chapter 1 we give an abstract codification of the kind of structure offered by our construction. There are three stages of the analysis. First we group together all points of M that "look alike" locally in the pair (Q, M). This expresses M as a disjoint union of open sets $\mathcal{M}_i$, each of which is a manifold and in fact a union of open simplexes of A. The family $\{\mathcal{M}_i\}$ is called a "variety" of M in Q. It follows that each closure cl $\mathcal{M}_i$ is a subcomplex A_i of A. Our construction thus provides us with regular neighborhoods of A_i in Q, in M and in each A_j (by Cohen's definition of a regular neighbourhood [7]). These regular neighbourhoods fit together tidily enough, and we describe the situation in terms of a "normal regular neighbourhood system" for the variety $\{\mathcal{M}_i\}$ of M in Q. Next we note that $\mathcal{M}_i$, in case Q is a manifold, is a submanifold of Q, and that our construction puts on (very roughly) the regular neighbourhood of A_i in Q the structure of a normal block bundle -- this is Rourke and Sanderson's construction. This suggests that the regular neighbourhoods of A_i in the general Q, M and the various A_j can also be given block bundle structures, if we allow block bundles whose fibre is no longer a disk but any cone. Our final step is to define such block bundles and describe how they are to fit together. This completes our definition of a "stratification" of M in Q. It should be observed that our construction for M in Q includes a construction for M in itself. Thus our final stratification $\underline{\xi}$ of M in Q includes a stratification $\underline{K}$ of M. It is fruitful to think of $\underline{\xi}$ as a "generalized bundle" over $\underline{K}$; this is why I use the notation $\underline{\xi}/\underline{K}$ with Greek letters for the "bundle" space and Roman letters for the "base" space.

In Chapter 2 we simply verify that the result of our construction satisfies the axiomatic systems of Chapter 1.

Theorems about stratifications are proved by dealing inductively with one block bundle at a time. (It is a meta-theorem that any geometrical proof in Rourke and Sanderson's [19, I and II] applies to a block bundle with arbitrary fibre whose base is a manifold. This idea is developed in the Appendix; see also p. vii.) In fact, over each $\mathcal{M}_i$, its block bundles in Q, M and so on, fit like sub-bundles one of another, and can be treated simultaneously. Difficulties arise when passing from block bundles over some $\mathcal{M}_i$ to block bundles over some other $\mathcal{M}_j$. In proving the uniqueness theorems of Chapter 3 for stratifications of fixed M in Q, these difficulties are overcome by a massive use of induction on "complexity" (that is, the number of terms in the variety). I should add that the uniqueness of normal regular neighbourhood systems of M in Q is a straightforward consequence of the usual uniqueness theorem for regular neighbourhoods. We also prove that if $\underline{K}$ is a fixed stratification of the variety $\{\mathcal{M}_i\}$ of M, then there is an essentially unique stratification $\underline{\xi}$ of M in Q which is over $\underline{K}$. This theorem reduces the description of abstract regular neighbourhoods of M to the description of abstract stratifications $\underline{\xi}$ over $\underline{K}$ - a generalization of the description of bundles over a given base space.

We have now generalized to stratifications the first main result about block bundles. In Chapter 4 we consider transversality, proving this theorem: Given polyhedra X, Y in a manifold M. Then there is an arbitrarily small isotopy of M which carries Y "block transverse" to a stratification of X in M. Further, if a subpolyhedron Y' of Y (which satisfies a certain natural condition) is already block transverse to the stratification, then we may keep Y' fixed during the isotopy. This theorem is new, I believe, even in case X, Y are submanifolds of M and Y' of Y. The difficulties I mentioned before prevent me from proving (or disproving) that block transversality is symmetric.

As an application of the relative transversality theorem, we prove in Chapter 5 an analogue of Thom's theorem [27] which exhibited an isomorphism between the group of cobordism classes of n-dimensional differentiable manifolds, and the n-th homotopy group of the spectrum of Thom complexes of classifying bundles for the orthogonal groups. One would like to define polyhedra X and Y to be "cobordant" if there is a polyhedron W whose boundary is the disjoint union of X and Y. However, every polyhedron X without boundary would then bound the cone cX; so this cobordism theory is trivial. We define non-trivial theories thus: Let $\mathfrak{F}$ be a family of polyhedra (satisfying certain conditions). A polyhedron X is an "$\mathfrak{F}$-polyhedron" if (roughly) for every point $x \in X$, its link $\ell k(x, X) \in \mathfrak{F}$. Now "$\mathfrak{F}$-theory" is the cobordism theory in which X, Y and W are all required to be $\mathfrak{F}$-polyhedra. An "$\mathfrak{F}$-classifying" stratification $\underline{Y}/\underline{\mathfrak{u}}$ is a stratification of an

$\mathfrak{F}$-polyhedron in a manifold which satisfies an appropriate universal property for "morphisms" of stratified $\mathfrak{F}$-polyhedra $\underline{K} \to \underline{U}$. By such morphisms into $\underline{U}$ are classified all stratifications over $\underline{K}$ in manifolds. Hence, in the light of Chapter 3, we can classify all abstract regular neighbourhoods (Q, M), where M is a fixed polyhedron and Q a (variable) manifold. If $\underline{\xi}/\underline{K}$ is a stratification of M in Q, then the "Thom space" of $\underline{\xi}$ is defined by identifying to a point the complement of a regular neighbourhood of M in Q. Then our analogue of Thom's theorem is: For each family $\mathfrak{F}$, there is an isomorphism χ : $\mathfrak{F}$-theory $\to$ the homotopy groups of the spectrum of Thom complexes of $\mathfrak{F}$-classifying stratifications.

As another consequence of the relative transversality theorem we prove: Let μ/K be a disk block bundle (in the sense of Rourke and Sanderson) with K any polyhedron. Let M be an abstract regular neighbourhood of K which is a manifold. By extending μ over M we obtain an abstract regular neighbourhood μ^*M of μ which is a manifold. Then the function μ^* : (Manifold regular neighbourhoods of K) $\to$ (Manifold regular neighbourhoods of μ) is a bijection.

In Chapter 6 this question is raised: given $X \subseteq Y$, when does X have a (disk)-block bundle neighbourhood in Y? A necessary condition is that X be "locally flat" in Y; that is, for every point $x \in X$, there are neighbourhoods U of x in X and V of x in Y such that $U \subseteq V$, and the pair (V, U) is p.ℓ. isomorphic to the pair $(U \times D, U \times 0)$, where D is a disk, 0 an internal point of D. However, this condition is not sufficient. A primary type of obstruction is defined and, in principle, classified. These obstructions are not sufficient; the usual difficulties arise. I have tried to elucidate the problem though I cannot solve it. It is related to the difficulty in proving block transversality symmetric.

In Chapter 7 I have listed various problems concerning stratified polyhedra that interest me; some spring directly from the material of the previous chapters; some are not yet related to this material and I think they should be.

In the Appendix is outlined the theory of block bundle flags. I have used an even more general setting, in order to deal simultaneously with other structures needed in this paper. The method of developing the theory, including the statements of results and their proofs, are largely a straightforward generalization of the corresponding parts of Rourke and Sanderson's [19, I and II]. Accordingly, I have only given those details of proof which might not be immediately apparent.

As this paper was being prepared for the press, I proved that block transversality between a polyhedron and a submanifold of a manifold is symmetric. I have inserted a proof as Chapter 8.

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