

Lecture Notes in Mathematics

A collection of informal reports and seminars

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**Random Integral Equations
with Applications to
Stochastic Systems**



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PREFACE

Over the past few years we have been engaged in research concerning random or stochastic integral equations and their applications. A general theory of random integral equations of the Volterra and Fredholm types has been developed utilizing the theory of "admissibility" of spaces of functions and fixed-point methods of probabilistic functional analysis. We have two main objectives in these notes. First, we wish to give a complete presentation of the theory of existence and uniqueness of random solutions of the most general random Volterra and Fredholm equations which have been studied heretofore. The second objective is to emphasize the application of our theory to stochastic systems which have not been extensively studied before this time due to the mathematical difficulties that arise.

These notes will be of value to mathematicians, probabilists, and engineers who are working in the area of systems theory or to those who are merely interested in the theory of random equations.

It is anticipated that we will expand these notes to include other types of stochastic integral equations, such as the Hammerstein type and Ito's equation, along with many other applications in the general areas of engineering, biology, chemistry, and physics. We hope to reach this goal by 1972.

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June, 1971

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VI. THE STOCHASTIC DIFFERENTIAL SYSTEMS

$$\dot{\mathbf{x}}(t; \omega) = A(\omega)\mathbf{x}(t; \omega) + b(\omega)\phi(\sigma(t; \omega))$$

WITH

$$\sigma(t; \omega) = \langle c(t; \omega), \mathbf{x}(t; \omega) \rangle$$

AND

$$\dot{\mathbf{x}}(t; \omega) = A(\omega)\mathbf{x}(t; \omega) + b(\omega)\phi(\sigma(t; \omega))$$

WITH

$$\sigma(t; \omega) = f(t; \omega) + \int_0^t \langle c(t-\tau; \omega), \mathbf{x}(\tau; \omega) \rangle d\tau130$$

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VII.THE STOCHASTIC DIFFERENTIAL SYSTEMS

$$\dot{x}(t; \omega) = A(\omega)x(t; \omega) + \int_0^t b(t-\eta; \omega) \phi(\sigma(\eta; \omega)) d\eta$$

WITH

$$\sigma(t; \omega) = f(t; \omega) + \int_0^t \langle c(t-\eta; \omega), x(\eta; \omega) \rangle d\eta$$

AND

$$\begin{aligned} \dot{x}(t; \omega) = A(\omega)x(t; \omega) + \int_0^t b(t-\tau; \omega) \phi(\sigma(\tau; \omega)) d\tau \\ + \int_0^t c(t-\tau; \omega) \sigma(\tau; \omega) d\tau \end{aligned}$$

WITH

$$\sigma(t; \omega) = f(t; \omega) + \int_0^t \langle d(t-\tau; \omega), x(\tau; \omega) \rangle d\tau144$$

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$$\dot{x}(t; \omega) = A(\omega)x(t; \omega) + B(\omega)x(t-\tau; \omega) + b(\omega) \phi(\sigma(t; \omega))$$

WITH

$$\sigma(t; \omega) = f(t; \omega) + \int_0^t \langle c(t-s; \omega), x(s; \omega) \rangle ds$$

AND

$$\begin{aligned} \dot{x}(t; \omega) = A(\omega)x(t; \omega) + B(\omega)x(t-\tau; \omega) \\ + \int_0^t \eta(t-u; \omega) \phi(\sigma(u; \omega)) du + b(\omega) \phi(\sigma(t; \omega)) \end{aligned}$$

WITH

$$\sigma(t; \omega) = f(t; \omega) + \int_0^t \langle c(u; \omega), x(t-u; \omega) \rangle du156$$

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