Lecture Notes in Mathematics

A collection of informal reports and seminars
Edited by A. Dold, Heidelberg and B. Eckmann, Zürich

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Lectures on n-Dimensional Quasiconformal Mappings

Springer-Verlag
Berlin · Heidelberg · New York 1971

AMS Subject Classifications (1970): 30 A 60

ISBN 3-540-05648-3 Springer-Verlag Berlin · Heidelberg · New York
ISBN 0-387-05648-3 Springer-Verlag New York · Heidelberg · Berlin

Offsetdruck: Julius Beltz, Hemsbach/Bergstr.

PREFACE

These notes are based on my lectures at the university of Helsinki in 1967-1968. They were first supposed to be published in another series, and a complete manuscript was given to the publisher in March 1969. When it turned out that the notes could not be published without a considerable delay, they were transferred to the Springer-Verlag in 1971. I have made only some small changes to the original manuscript and added references to the newest literature.

I wish to express my sincere thanks to F. W. Gehring, J. Hesse, R. Näkki, and S. Rickman, who read the manuscript and made valuable suggestions.

Helsinki
July 1971

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INTRODUCTION

By a classical theorem of Liouville, every conformal mapping of a domain in the euclidean n-space R^n, $n \geq 3$, is a restriction of a Möbius transformation, that is, member of the group generated by similarity mappings and inversions in spheres. For this reason, the theory of conformal mappings is essentially 2-dimensional. The situation is different with quasiconformal mappings. Let us consider a diffeomorphism f of a domain $D \subset R^n$ onto a domain $D' \subset R^n$. The derivative of f at a point $x \in D$ is a bijective linear mapping $f'(x) : R^n \to R^n$. The diffeomorphism f is called quasiconformal if the ratio

$$H(x, f) = \frac{\max\limits_{|h|=1} |f'(x)h|}{\min\limits_{|h|=1} |f'(x)h|}$$

is bounded in D. Usually it is more convenient to use a more general definition in which f is not required to be everywhere differentiable. However, it is easy to see that there are plenty of quasiconformal mappings in R^n. For example, if $f : D \to D'$ is a diffeomorphism and if D_0 is a domain whose closure is a compact subset of D, then the restriction $f | D_0$ is quasiconformal. Furthermore, while the conformal image of a ball is always a ball or a half space, it is possible to construct a quasiconformal mapping of a ball onto a domain which has non-accessible boundary points (Gehring-Väisälä [2, p. 60]).

2-dimensional quasiconformal mappings were introduced by Grötzsch [1] in 1928. A rather comprehensive treatment of the present state of the theory is given in the excellent books of Ahlfors [3] and Lehto-Virtanen [1]. Higher dimensional quasiconformal mappings were first considered by Soviet mathematicians Lavrentiev [1], Markušević [1] and Kreines [1] in 1938-1941, but the theory was practically forgotten for 18 years. Since 1959, however, the n-dimensional quasiconformal mappings have been studied rather extensively by a great number of authors in several countries.

The purpose of these notes is to give an exposition of the basic theory of quasiconformal mappings in R^n. The aforementioned books of Ahlfors and Lehto-Virtanen give the historical background, although no previous knowledge is needed on 2-dimensional quasiconformal mappings. In fact, our proofs apply also to the case $n = 2$. However, in this case the proofs could often be simplified, thanks to the Riemann mapping theorem.

We assume that the reader is familiar with the basic facts of the theory of measure and integration. More advanced results of real analysis are given in Chapter 3. Almost all what is needed and much that is included, is contained in the books of Munroe [1] and Saks [1].

We also assume some knowledge on the topology of euclidean spaces. The required facts can be found in the books of Newman [1] and Wilder [1, pp. 51-68]. We shall use the phrase "by Topology" when we are appealing to a topological result (such as the invariance of domain) which is intuitively obvious but often rather profound.

Two important topics have been omitted. We do not prove the theorem of Gehring and Rešetnjak, which states that every 1-quasiconformal mapping is a Möbius transformation for $n \geq 3$. Neither do we present Gehring's theory on the symmetrization of rings. This seems to be unavoidable when deriving sharp bounds in certain modulus estimates. Our results are, therefore, often qualitative rather than quantitative.

The quasiconformal mappings form a subclass of the class of quasiregular mappings, which are not necessarily homeomorphisms. This larger class has not been systematically studied until since

1966, and it is not considered in these notes.

References and brief historical remarks are given at the ends of the sections. The bibliography contains only the publications which are referred to in the text. A very comprehensive bibliography is given in the monograph of Caraman [1].

NOTATION AND TERMINOLOGY

N = the set of positive integers.

Z = the set of integers.

R^1 = the set of real numbers.

R^n = the n-dimensional euclidean space. We identify R^{n-1} with the subspace $x_n = 0$ of R^n.

The letter n denotes always the dimension of the space in question.

$e_1, \ldots, e_n$ = the coordinate unit vectors of R^n. For example, $e_1 = (1, 0, \ldots, 0)$.

The coordinates of a point $x \in R^n$ are denoted by $x_1, \ldots, x_n$. Thus $x = x_1 e_1 + \ldots + x_n e_n$. However, we use subscripts also as indices if there is no danger of misunderstanding. For example, a sequence of points in R^n is often denoted by $x_1, x_2, \ldots$ or by (x_j). The norm of a vector $x \in R^n$ is written as

$$|x| = (x_1^2 + \ldots + x_n^2)^{1/2} .$$

$B^n(x_0, r)$ is the ball $\{x \in R^n \mid |x-x_0| < r\}$. $B^n(r) = B^n(0, r)$. $B^n = B^n(0, 1)$.

$S^{n-1}(x_0, r)$ is the sphere $\{x \in R^n \mid |x-x_0| = r\}$. $S^{n-1}(r) = S^{n-1}(0, r)$. $S^{n-1} = S^{n-1}(0, 1)$. The dimension $n-1$ is sometimes omitted.

$\bar{R}^n = R^n \cup \{\infty\}$ = the one point compactification of R^n. Thus $\bar{R}^n$

is a topological space, homeomorphic to S^n. A metric in $\bar{R}^n$ will be defined in Section 12.

$\dot{R}^1$ = the two-point compactification $R^1 \cup \{-\infty, \infty\}$ of R^1.

Let $A \subset \bar{R}^n$. $\bar{A}$ is the closure of A. ∂A is the boundary of A. $\operatorname{int} A$ is the interior of A. $\underline{C}A$ is the complement of A. All these are taken with respect to $\bar{R}^n$. This justifies the notation $\bar{R}^n$.

By a ball neighborhood of a point $x_0 \in \bar{R}^n$ we mean a ball $B^n(x_0, r)$ if $x_0 \neq \infty$ and a set $\underline{C}B^n(r)$ if $x_0 = \infty$.

The set-theoretical difference of two sets A and B is denoted by $A \setminus B = \{x \mid x \in A, \ x \notin B\}$.

Since points of R^n are treated as vectors, we use the group-theoretic notation $A + B = \{a + b \mid a \in A, b \in B\}$ if A and B are subsets of R^n. Similarly, we define the sets $A - B$, $x + A$, rA, etc., where $x \in R^n$ and $r \in R^1$. $d(A, B)$ is the distance between A and B, and $d(A)$ is the diameter of A.

Let $a, b \in \dot{R}^1$, $a \leq b$. Then $[a, b]$ is the closed interval $\{t \mid a \leq t \leq b\}$. If $a < b$, (a, b) is the open interval $\{t \mid a < t < b\}$. A closed (open) n-interval is the cartesian product of n closed (open) intervals of R^1.

A neighborhood of a point or a set is an open set containing it. A domain is a connected non-empty set.

The notation $f : D \to D'$ includes the assumption that D and D' are domains in $\bar{R}^n$. If Γ is a curve family in D, then Γ' denotes always its image under f.

Let U be an open set in R^n. A mapping $f : U \to R^m$ is differentiable at $x \in U$ if there is a linear mapping $f'(x) : R^n \to R^m$, called the derivative of f at x, such that

$$f(x + h) = f(x) + f'(x)h + |h| \varepsilon(x, h)$$

where $\varepsilon(x, h) \to 0$ as $h \to 0$. The jacobian of f at x is denoted by $J(x, f)$.

If $A \subset R^n$, $m_n^*(A)$ is the Lebesgue outer measure of A. $m_n^*(A)$

is also defined if A is a subset of a given n-dimensional linear submanifold or of a given n-dimensional sphere in $R^{n'}$, $n' > n$. The subscript n may be omitted if there is no danger of misunderstanding. The measure of a set $A \subset \bar{R}^n$ is defined as the measure of $A \setminus \{\infty\}$.

$\Lambda^*_\alpha(A)$ is the α-dimensional Hausdorff outer measure of A, defined in Section 30. The star is omitted if A is measurable.

The integral of a function $f : A \to \dot{R}^1$ over a set $E \subset A$ is denoted by

$$\int_E f \, dm_n \qquad \text{or} \qquad \int_E f(x) \, dm_n(x) .$$

It is defined if E and f are m_n-measurable and if either f is non-negative or $\int_E |f| \, dm_n < \infty$. In the first case, the integral may have the value ∞. In the second case, f is called integrable over E. The subscript n may again be omitted. Also E can be omitted if $E = R^n$.

The class of Borel sets in a topological space is the smallest σ-algebra which contains the open sets. If A is a Borel set and if T is a topological space, a mapping $f : A \to T$ is said to be a Borel function if $f^{-1}U$ is a Borel set for every open set U in T.

$\Omega_n = m_n(B^n)$ and $\omega_n = m_n(S^n)$. Explicitly,

$$\omega_{n-1} = n\,\Omega_n , \quad \omega_{2k-1} = \frac{2\pi^k}{(k-1)!} , \quad \omega_{2k} = \frac{2^{k+1}\,\pi^k}{1\cdot 3 \cdots (2k-1)} .$$

C^k = the class of k times continuously differentiable mappings.

L^p = the class of functions f such that $|f|^p$ is integrable.

If $A : R^n \to R^n$ is a linear mapping, then

$$|A| = \max_{|h|=1} |Ah| , \quad \ell(A) = \min_{|h|=1} |Ah| ,$$

and $\det A$ is the determinant of A.

The words "increasing" and "decreasing" are used in the weak sense. For example, a function $f : (a , b) \to R^1$ is increasing if

$a < s < t < b$ implies $f(s) \leq f(t)$.

iff = if and only if.

qc = quasiconformal.

qclly = quasiconformally.

qcty = quasiconformality.

$\Delta\!\!\Delta$ = the end of a proof.

We give a list for other notations, which will be defined in the text and used throughout the rest of the notes.

$\ell(\alpha)$ length of a path 1

$|\alpha|$ locus of a path 1

s_α length function 2

$L(x, f)$ 11

$F(\Gamma)$ 16

$M_p(\Gamma)$, $M(\Gamma)$ modulus 16

$\Gamma_2 > \Gamma_1$ 17

$\Delta(E, F, G)$ 21

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$M_p^S(\Gamma)$ modulus on a manifold 28

b_n constant 28

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$R(C_o, C_1) = \underline{C}(C_o \cup C_1)$ 33

$\varkappa_n(r)$ 34

$q(a, b)$ spherical distance 37

$\lambda_n(r)$ 38

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$\operatorname{ker}_{j\to\infty} E_j$ kernel of a sequence of sets 73

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