Lecture Notes in Mathematics

A collection of informal reports and seminars
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Foundations of the Theory of Klein Surfaces

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INTRODUCTION

It has long been known that the category $\textcircled{S}$ of compact Riemann surfaces and non-constant analytic maps, and the category $\textcircled{C}$ of complex-algebraic function fields and complex isomorphisms are, via two contravariant functions, coequivalent; thus an analytic theory and an algebraic theory are tied together.

While investigating several Banach algebras on compact Riemann surfaces $\mathfrak{X}$ with non-empty boundary ∂X ([A_2], [A_3], [A_4]), the first author posed the following question for himself: what is the simplest algebraic object which can be associated with $\mathfrak{X}$, from which $\mathfrak{X}$ can be recovered? The answer seems to be the following: let $E(\mathfrak{X})$ be the field of all functions f meromorphic on $\mathfrak{X}$ such that $f(\partial X) \subset R \cup \{\infty\}$. This field is an algebraic function field in one variable over the reals. It is natural then to ask the converse question: given such a field E, is there a compact Riemann surface $\mathfrak{X}$ (possibly with boundary) such that $E = E(\mathfrak{X})$? The answer to this question, interestingly, is no. The following field, long known to algebraic geometers, supplies a counter-example to such a conjecture: let $E = R(x,y)$, where $x^2 + y^2 = -1$. The present collaboration began at this juncture.

Let $\textcircled{R}$ be the category of all real-algebraic function fields and all real-linear isomorphisms. Given such a field E, the algebraic geometers have long known how to associate a curve X with E; for example let $X \equiv \{\mathfrak{O} : \mathfrak{O}$ a valuation ring of E over $R\}$. The usual topology put on such a curve is the Zariski topology in which the proper closed sets are the finite sets. Such a topology does not utilize the topology on R and does not render X a manifold. It is relatively easy to define another topology on X, utilizing the topology of R, under which X is a compact surface (possibly with boundary). At this point in our investigations, we hypothesized that algebraic "functions" on such a manifold must be "analytic," in some sense. Our task was then to find the correct "analytic" structure on X.

In the example given above, X is the real projective plane; thus non-orientable X arise. Since conformal maps are orientation-preserving, the definition of "analytic" structure needed to be relaxed so that non-orientability could occur. The way out of this is to allow the transition functions t to be analytic (i.e., $\partial t/\partial \bar{z} = 0$) or anti-analytic (i.e., $\partial t/\partial z = 0$): that is dianalytic. Having roughed out this dianalytic theory, we discovered that Schiffer and Spencer [SS] had, not surprisingly, done it before us; but -- being less interested in the compact case -- had not come to a full realization of the algebraic consequences of this dianalytic theory. Further, they do not define morphisms and functions directly for non-orientable surfaces. Going back to an old tradition [B], we have chosen to call surfaces with dianalytic structure Klein surfaces. Much of our Chapter 1 lays the foundations for the intrinsic study of morphisms, functions, and differentials on Klein surfaces.

A brief summary of the contents is as follows: Sections 1.1 - 1.4 contain the basic definitions of Klein surfaces, meromorphic functions, and morphisms. The main novelty here is that if $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ is a morphism then X (the underlying space of $\mathfrak{X}$) may be folded over ∂Y. Sections 1.5 - 1.9 deal with methods of putting dianalytic structure on surfaces: particularly lifting and descending dianalytic structure under covering maps. It is shown that every compact surface can carry dianalytic structure. The dianalytic structures on the disc, real projective plane, annulus, Klein bottle, and Möbius strip are classified. Section 1.10 treats the theory of differentials and their integration on Klein surfaces, while 1.11 deals with automorphisms of Klein surfaces. Most of Chapter 1 is devoted to the general theory of Klein surfaces; in Chapter 2 we deal with the compact case. The main theorem is the following: the category of compact Klein surfaces and the category of algebraic function fields in one variable over R are co-equivalent. That is, compact Klein surfaces can be considered as non-singular algebraic curves over the reals. Various applications of this theorem are given. A research announcement summarizing most of the results of this monograph has appeared [AG].

Our aim in writing these lecture notes is to present a fairly complete account of a recently re-vitalized theory, accessible to non-experts. This is not to say that no background is required; it is. On the analytic side some knowledge of the theory of Riemann surfaces will be an enormous help to the reader. On the algebraic side, some knowledge of algebraic curves and their function fields is required to appreciate Chapter 2. Even with these pre-requisites, we have tried to ease the task of the reader not steeped in these classical theories by supplying some review of known facts. For example, the first section of Chapter 2 is entirely review.

The history of this subject is long, going back to Klein's 1882 Monograph on Riemann's theory of Abelian integrals [K], for -- in the closing pages -- he considers the group of conformal maps of the Klein bottle, and other non-orientable surfaces. No definition, acceptable to a modern reader, seems to have been known before the appearance, in 1913, of Hermann Weyl's definitive work on Riemann surfaces [W_2]. At that time, and for some time thereafter, all surfaces under consideration are assumed to be orientable. Schiffer and Spencer return to Klein's concept in their 1954 monograph [SS] and thus give the non-orientable case its first modern treatment. On the algebraic side, L. Berzolari [B] gave an account on plain algebraic curves, at about the turn of the century, which did treat curves over the reals. However his work was severely handicapped by the lack of valuation theory, which was not then used well enough to treat the real case with ease. By that time algebraic geometry had generalized itself to a very great extent and had left real-algebraic curves far behind.

The fact that any real-algebraic function field E, whose constant field is R, is just the fixed field of $F \equiv E(i)$ under an R-linear automorphism σ of order two, and thus that the Klein surface $\mathfrak{X}$ of E is just the orbit space of the Riemann surface $\mathfrak{W}$ of F under σ^* -- is obvious, and was well known. Our thesis has been that it is better to work with E on $\mathfrak{x}$ rather than to have to work with E imbedded in $E(i)$ on $\mathfrak{B}$. Hopefully this monograph and successive work in this direction will give weight to our point of view. Already application has been made by the first author to the theory of real-Banach algebras [A_5]. These preliminary remarks were elaborated by the first author and L. Andrew Campbell in [AC] to prove an extension of the Arens-Royden Theorem to real Banach algebras. Another offshoot of this theory, that of analytic and harmonic obstruction on non-compact Klein surfaces, has been considered by the first author [A_6]. In [A_7] he is making a study of algebraic analytic obstruction theory on compact Klein surfaces and the way it reflects the topological obstruction on the underlying surface. The second author has dealt with the foundations of sheaf theory on Klein surfaces [G_1]; and he, together with Walter Read have investigated the question of the existence of positive differentials on elliptic or hyperelliptic Klein surfaces with boundary [GR]. Further work is contemplated.

Many mathematicians aided us in our bibliographic researches. Professors Ahlfors and Bers and Sir Edward Collingwood were particularly helpful. We would also like to thank our colleague Professor Sanford Segal (of Rochester) for bringing the paper by Berzolari to our attention.

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