Lecture Notes in Mathematics

A collection of informal reports and seminars
Edited by A. Dold, Heidelberg and B. Eckmann, Zürich

145

Eduardo J. Dubuc

University of Chicago, Chicago, IL/USA

Kan Extensions in Enriched Category Theory

Springer-Verlag
Berlin · Heidelberg · New York 1970

Title No. 3302

P R E F A C E

Category Theory is rapidly coming of age as a Mathematical discipline. In this process it now appears that a central role will be played by the notion of an enriched category. These categories, with hom-sets in a closed category,--in particular, in a symmetric monoidal closed category--seemed initially very complex and difficult to manage effectively. However, independent work by various experts--Yoneda, Linton, Bénabou, Eilenberg-Kelly, Lambek, Bunge, Ulmer, Gray, Palmquist, and others--has considerably improved the situation. A vital step was the discovery of the proper use of tensors, cotensors, and Kan extensions for enriched categories (A discovery made simultaneously and independently by Bénabou and by Kelly with Day). As a result, an efficient presentation of enriched categories is now possible.

This paper by Dubuc collects all these ideas in a compact exposition which makes this efficiency very clear--and which also serves as a basis for Dubuc's own original contributions. I have, therefore, recommended to the editors of the _Lecture Notes_ series the rapid publication of this paper, to provide easy access to this foundation for future development.

Saunders Mac Lane

TABLE OF CONTENTS

INTRODUCTION

The original purpose of this paper was to provide suitable enriched completions of small enriched categories. We choose as a precise meaning for the words "enriched category" the notion of V-category, where $\mathbb{V}$ is any given (fixed) symmetrical monoidal closed category ([2]), abbreviated closed category (see [3], introduction). In introducing the notation we recall the above notions, but the reader so inclined can perfectly well go through this paper thinking "category", "functor" every time he sees the words "V-category", "V-functor", (and etc.).

We found it necessary to set up an appropriate background in which facts about an enriched world are stated, and in doing so we (comfortably) put ourselves in a non-autonomous treatment whose basic guidelines can be subsumed in the questions: "From which minimal set of basic facts about the set-based world can we deduce the existence of completions of small categories?", "Which of those facts, literally or suitable translated, are still true in the V-world, and what conditions should be imposed on $\mathbb{V}$ in order to rescue all of them?" It is clear that given any result of set-based category theory these two questions can be used as guidelines for an investigation of the enriched worlds. We hope that in this paper it is shown that the result concerning the existence of completions for a small category, rich enough in different notions but yet simple, leads to sufficient knowledge of the V-world of nature, knowledge which is indispensible to achieve the desirable formulation and development of an appropriate autonomous (axiomatic) foundation of the Category of V-categories (and of $\mathbb{V}$ itself), in a way similar to the one developed in the pioneer work of Lawvere for the ordinary set-based world. We also hope that the techniques employed in this paper, techniques that rest on an intensive use of Kan extensions and which are perfectly suited for such an autonomous treatment, have as well (because of their simplicity and economy) their own interest even when applied to obtain the known ordinary results, bringing a better understanding of the reasons why those results hold. On the other hand, due to the proliferation of closed categories interesting for mathematical practice (see for example Bunge [7] where a list of some of them is provided) this paper should provide a common setting for many different constructions and results in mathematics.

In order to have a handy reference in pre-section 0 we took from Kelly [3] the basic results concerning V-adjunctions. Talking about only one side of the duality, the unique completeness concept (right Kan extension) of ordinary set-based category splits into four different (and independent) ones in the V-context. This is a clear imperfection of this non-autonomous treatment, but a careful reading of the paper seems to indicate that only two of them are essentially needed, namely, the one of cotensor and the one of right Kan extension . In I.1, I.2 and I.3 we introduce V-limits, cotensors and ends, and most of the definitions and results were taken from the summary papers of Day-Kelly [1], [3], except for our careful treatment of the concepts applied to V-functors, where a distinction appears between V-functors which satisfy the universal property and V-functors which pointwise satisfy the universal property. The later ones are characterized by the fact that the universal property is preserved by the representable functors of the codomain category. This treatment also serves to fill a gap in ordinary category theory, that, if probably known by many authors, has to my best knowledge never been written down on paper. In I.4 we make a parallel treatment of right Kan extensions, where we set the properties which enable us to use Kan extensions as the single major tool through all the paper. We also give a formal criterion for the existence of V-left adjoint (Benabou [6]) and the V-version of the well known classical Kan formula, (which we took from [3], where a proof of a stronger result is given which applies only in the small and complete case).

We thought that the best way of introducing the relevant concepts of generator, generating functor and dense functor, as well as the best way to do the necessary constructions needed in the completion of a small category (Lambeck [4]) was by means of the use of monads (triples). This technique has also the advantage that it can be generalized to the V-context without any further complications. Central is the concept of codensity monad, which we took from Linton [9], and which we introduce here as a particular right Kan extension.

We found it also necessary to use a fair number of properties of monads in the V-context (V-monads), and so in Chapter II we put Kan extensions to work in order to develop that part of the theory of V-monads that we use later in Chapter III and IV.

It should be noticed that Chapter II (as well as the rest of the paper) is written without recourse to ordinary set-based results, and thus, when applied to this case, it provides new techniques (different proofs) for achieving these results. These techniques should be called "Kan extensions techniques". For example; in II.1, we observe that the general Semantics-Structure adjointness is just the Kan extension universal property of the codensity V-monad.

In III.1 we develop general properties of V-continuous V-functors; and in doing so we have follow as a basic guideline the paper of Lambeck [4], some of whose propositions are here literally translated into the V-context. In III.2 we assume for the first time (except for the case of cotensors) the existence of completeness concepts in the categories that we work with, and we develop the special properties of V-complete V-categories. Central in this section is the V-Special Adjoint Functor Theorem, which we deduce as a corollary from properties of Kan extensions. The proof of this theorem suggests that the relevant property of cogenerators is not that of producing solution sets, but the fact that just by definition the unit of the associated codensity monad is a monomorphism. For example, in the V-context, a V-cogenerator will in general not be a real cogenerator, (the category $\mathbb{1}$ is a $\mathbb{C}$at-generator for $\mathbb{C}$at), but the special Adjoint Functor Theorem still holds.

In III.3 the concept of V-completion is introduced, and what we mean exactly by completion is explained. This has also been taken from Lambeck [4]. The reader interested in the problem of completions should (of course) consult the (fundamental) work of Isbell, that, because of its different language we have found difficult to introduce (or refer to) here. We give two different methods of constructing a V-completion for a small V-category. The first is just an adequate V-version of the completion obtained in Lambeck [4], and in order to achieve it we use a technique based on a straightforward use of V-monads. The second use the construction of a tower of categories and functors by means of limits of (small) chains of categories obtained as a result of an iterated construction of categories of algebras over a monad. The author first learned of this construction in a lecture given by Tierney in the Midwest Category Seminar at Urbana Illinois, where Tierney also suggested its possible usefulness in the construction of completions. After that lecture the author and Tierney himself discovered independently the crucial fact that the whole tower (a large chain of categories) has a limit which is a locally small category, and that the corresponding monad in that limit is the identity. Here we present a V-version of all these facts and we use them to provide a V-completion.

In IV.1 the construction of the V-category of V-functors and V-natural transformations is made in exactly the same way as in Day-Kelly [1], and in IV.2 we use the V-Yoneda embeddings as the starging data upon which we apply the process of construction of V-completions developed in III.3. Finally in IV.3 we take the definitions of corealization and cosingular functors from Applegate, Tierney [5], we develop the additional features of the tower constructed in III.3 under the presence of V-functor categories and we give a comparison of the two completions.

An appendix is given where we find conditions on a closed category $\mathcal{V}$ which imply that in the V-world cotensors in V-categories with limits are real limits.

The author hopes that it will not be totally incorrect to say that this paper is a testimony of two basic mathematico-philosophical principles. First, "the relevant properties of mathematical objects are those which can be stated in terms of their abstract structure rather than in terms of the elements which the objects were thought to be made of (Lawvere)" coupled with "the relevant facts of category theory hold because of formal interconnections between the concepts involved rather than because of their substantial content (which is none)".

This, because of that peculiar characteristic of the mind which leads every human being to the convinction that abstract ideas are real, can be pushed forward (extrapolated) into a simple purely philosophical principle, namely, "substance is form". Second, "everything in mathematics that can be categorized is trivial (Freyd)" which should be understood. "Category Theory is good ideas rather than complicated techniques."

* * *

We omit in the text all remarks concerning the uniqueness up to isomorphisms of concepts defined by means of universal properties, thus this fact should be present in the mind of the reader, especially because of our repeated use of the article "the" instead of the article "a".

We often denote isomorphisms by the same letter in both directions (especially in the case of adjunctions).

There are three kinds of statements in this paper, the ones which hold without any assumptions in $\mathcal{V}$ are headed in the usual way (Proposition ...). The ones which hold only when $\mathcal{V}$ has equalizers are headed with one black "•" preceeding them (• Proposition ...). The ones which hold only when $\mathcal{V}$ is a complete category (i.e., when it has (small) limits) are headed with two black "•" preceeding them (•• Proposition...).

Besides the few simplifying conventions mentioned above, we believe that other peculiarities that may have escaped the attention of the author will not lead to any confusion for the reader.

The rest of the notation is introduced in the text.

* * *

My thanks to M. Tierney and Ross Street for many conversations about the material presented in this paper, my special thanks to Professor A. P. Calderon who was responsible for my coming to the University of Chicago, and, my heartful thanks to Professor Saunders MacLane for help, mathematical and non-mathematical, without which this paper could not have been written. For his continued teaching, criticism and encouragement the value of my close contact with him during the past three years is impossible to overestimate.

E. J. D. December 1969

Terminology

A category $\mathbb{V}$ is monoidal if it has an associative tensor product $\mathbb{V} \times \mathbb{V} \xrightarrow{\otimes} \mathbb{V}$ with a unit $I \in \mathbb{V}$ (that is, $I\otimes - \approx$ id and $-\otimes I \approx$ id) and coherence.

A monoidal category $\mathbb{V}$ is symmetric if for every $V, W \in \mathbb{V}$, $V \otimes W \approx W \otimes V$, (natural) and coherence.

We call the functor $\mathbb{V} \xrightarrow{\mathbb{V}_o(I,-)} \mathbb{S}$ the base functor.

If $\mathbb{V}$ is a monoidal category, a V-category $\mathbb{A}$ is a class of objects A, B, C, ..., for any two A, B, a V-object "between" them, $\mathbb{A}(A,B) \in \mathbb{V}$, for any three A, B, C a "composition", that is, a map $\mathbb{A}(A,B) \otimes \mathbb{A}(B,C) \xrightarrow{o} \mathbb{A}(A,C)$ and for any A an "identity", that is, a map $I \xrightarrow{i} \mathbb{A}(A,A)$ in $\mathbb{V}$. This data is subject to the requirement that "o" be associative and "i" be a unit for "o".

If $\mathbb{A}$ is a V-category, a category (with the same class of objects) is obtained by defining $\mathbb{A}_o(A,B) \in \mathbb{S}$, $\mathbb{A}_o(A,B) = \mathbb{V}_o(I, \mathbb{A}(A,B))$. This category is usually called the "underlying" category of $\mathbb{A}$. By an abuse of language we consider $\mathbb{A}$ to be at the same time a V-category and a category. Thus, we use only one notational symbol, $\mathbb{A}$. $\mathbb{A}_o(AB)$ denotes the set of morphisms in $\mathbb{A}$ between A and B.

$\mathbb{A}(A,B)$ becomes a functor $\mathbb{A}^{op} \times \mathbb{A} \xrightarrow{\mathbb{A}(-,-)} \mathbb{V}$, $(A,B) \rightsquigarrow \mathbb{A}(A,B)$, and so for every (fixed) $A \in \mathbb{A}$ there is a functor $\mathbb{A} \xrightarrow{\mathbb{A}(A,-)} \mathbb{V}$ whose action on a morphism $B \xrightarrow{f} B'$ we often denote by

$\mathbb{A}(A,B) \xrightarrow{\mathbb{A}(\square,f)} \mathbb{A}(A,B')$ and dually

A V-functor between two V-categories; $\mathbb{A} \xrightarrow{F} \mathbb{B}$ is a function from the class of objects of $\mathbb{A}$ to the class of objects of $\mathbb{B}$, and for any two $A,B \in \mathbb{A}$ a map $\mathbb{A}(A,B) \xrightarrow{F_{AB}} \mathbb{B}(FA, FB)$ in $\mathbb{V}$, preserving the units and the composition.

If $\mathbb{A} \xrightarrow{F} \mathbb{B}$ is a V-functor, a functor, which we also denote by $\mathbb{A} \xrightarrow{F} \mathbb{B}$ is obtained by defining

$$\mathbb{A}_o(A,B) = \mathbb{V}_o(I,\mathbb{A}(AB)) \xrightarrow{\mathbb{V}_o(I,F_{AB})} \mathbb{V}_o(I,\mathbb{B}(FA,FB)) = \mathbb{B}_o(FA,FB) .$$

This functor is usually called the "underlying" functor of F. By an abuse of language, we consider F to be at the same time a V-functor and a functor.

Since $\mathbb{V}_o(I, F_{AB}) = \mathbb{V}_o(I, G_{AB})$ does not imply $F_{AB} = G_{AB}$, it is clear that different V-functors can be equal as functors, except when I is a generator, that is, when the base functor is faithful.

A V-natural transformations between two V-functors $F \xRightarrow{\varphi} G$ is a family of maps $FA \xrightarrow{\varphi A} GA$ such that for every pair of objects the diagram:

$$\begin{array}{ccc}
\mathbb{A}(A,B) & \xrightarrow{\quad F \quad} & \mathbb{B}(FA,FB) \\
\Big\downarrow {\scriptstyle G} & & \Big\downarrow {\scriptstyle \mathbb{B}(\square,\varphi B)} \\
\mathbb{B}(GA,GB) & \xrightarrow{\ B(\varphi A,\square)\ } & \mathbb{B}(FA,GB)
\end{array}$$

commutes

A V-natural transformation is a natural transformation but not vice-versa, except when I is a generator.

A <u>closed category $\mathbb{V}$</u> is a symmetrical monoidal category $\mathbb{V}$ such that for any object $V \in \mathbb{V}$ the functor $\mathbb{V} \xrightarrow{-\otimes V} \mathbb{V}$ has a right adjoint $\mathbb{V} \xrightarrow{\mathbb{V}(V,-)} \mathbb{V}$.

By the aid of the adjunction isomorphism maps can be defined $\mathbb{V}(V,W) \otimes \mathbb{V}(W,X) \longrightarrow \mathbb{V}(V,X)$ and $I \longrightarrow \mathbb{V}(V,V)$ making of $\mathbb{V}$ a V-category. So there is a functor $\mathbb{V}^{op} \times \mathbb{V} \xrightarrow{\mathbb{V}(-,-)} \mathbb{V}$ and for every V, the functor $\mathbb{V}(-,V)$ is adjoint on the right to itself.

For any V-category $\mathbb{A}$ and object $A \in \mathbb{A}$ the functors $\mathbb{A} \xrightarrow{\mathbb{A}(A,-)} \mathbb{V}$ and $\mathbb{A}^{op} \xrightarrow{\mathbb{A}(-,A)} \mathbb{V}$ become V-functors and there is a V-functor $\mathbb{A}^{op} \otimes \mathbb{A} \xrightarrow{\mathbb{A}(-,-)} \mathbb{V}$ (where $\mathbb{A}^{op} \otimes \mathbb{A}$ is the V-category whose class of objects is the cartesian product (Objects of $\mathbb{A}$) x (Objects of $\mathbb{A}$) with V-structure given by: $\mathbb{A}^{op} \otimes \mathbb{A}((A,B),(A',B')) = \mathbb{A}(A',A) \otimes \mathbb{A}(B,B')$. There is a (canonical) functor $\mathbb{A}^{op} \times \mathbb{A} \longrightarrow \mathbb{A}^{op} \otimes \mathbb{A}$.

When $\mathbb{A} = \mathbb{V}$, for every $V \in \mathbb{V}$ the V-functor $\mathbb{V} \xrightarrow{\mathbb{V}(V,-)} \mathbb{V}$ is V-right adjoint to the V-functor $\mathbb{V} \xrightarrow{-\otimes V} \mathbb{V}$ and the V-functor $\mathbb{V}(-,V)$ is V-adjoint on the right with itself (for the definition of V-adjointness see 0)

A morphism in $\mathbb{V}$ which is both a monomorphism and an epimorphism would not be in general an isomorphism. Hence V-full and V-faithful (clear definition) V-functors need not be such that the maps $\mathbb{A}(A,B) \xrightarrow{F_{AB}} \mathbb{B}(FA,FB)$ are isomorphisms. A V-functor such that F_{AB} is an isomorphism is called V-full-and-faithful.