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The Duality of Compact Semigroups and C*-Bigebras

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We propose a complete duality theory for the category of compact Hausdorff topological semigroups in terms of commutative C*-bigebras with identity. A bigebra (formerly called hyperalgebra or Hopf algebra) is an algebra with a comultiplication. In a more expository introductory part, we discuss the concept of a bigebra over a category, which plays an important role in this context. The second, more technical part starts with a discussion of C*-algebras and their tensor products. Finally, after the establishment of the duality, we show how it generalizes the Tannaka duality theory for compact groups, and the Pontryagin duality for compact abelian groups.

0. Introduction

One of the best known duality theories is the one between compact abelian groups and discrete abelian groups, discovered by PONTRYAGIN in the early thirties. A duality theory for not necessarily commutative compact groups was later described by TANNAKA, and its most modern forms are comparatively recent and go back to HOCHSCHILD [8].

All of these duality theories are closely tied up with the representation theory of these groups. At first sight this observation makes the outlook to possible generalizations for wider classes of compact objects (compact semigroups, say) seem gloomy. It is indeed well known and has been accepted with pessimistic fatalism that no useful representation theory is available for the category of all compact semigroups. The reason of this default is the absence in general compact semigroups of an invariant integral whose support is the entire semigroup. (In fact the only compact semigroups having a two sided invariant integral with full support are the groups.)

On the other hand, if one proceeds to the end of the spectrum of compact objects, namely those objects having no additional structure whatsoever, then again we have a completely satisfactory duality theory: namely, the duality between compact spaces and commutative C*-algebras which is based on the theorem of GELFAND and NAIMARK. It is therefore natural to look for dual objects of compact semigroups in the class of commutative C*-algebras with an additional element of structure. And indeed we will be able to establish a complete duality between the category of compact semigroups and the category of C*-algebras with a comultiplication. Algebras of certain types which, in addition, are endowed with a comultiplication have traditionally been called hyperalgebras by the algebraic geometers [2] and Hopf algebras by the algebraic topologists. HOPF observed there occurrence first in the cohomology ring of compact Lie

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groups (we will briefly outline this example in our discourse). Authors like DIEUDONNÉ and CARTIER (see e.g. [2]) used hyperalgebras extensively in the theory of formal Lie groups and algebras. Yet we will follow the more recent terminology which is sanctified by BOURBAKI. He calls an object without a multiplication but with a comultiplication a cogebra, and an object with a multiplication and a comultiplication a bigebra. Since we operate on C*-algebras we call a C*-algebra with a comultiplication in most cases a C*-bigebra.

The first part of our discussion is of a more narrative nature and describes the categorical background which we will make quite concrete by pointing out many relevant examples.

The brief Section 1 is devoted to a definition of duality, Section 2 to the idea of a multiplicative category.

In Sections 3, 4, 5 we discuss the concept of a bigebra in the spirit of category theory by starting with algebras, then proceeding to cogebras (the dual concept, and then finally by synthesizing the two to the concept of a bigebra. This part of the discussion is enriched by examples in order that the reader who considers a bigebra an unfamiliar if not useless object may get accustomed to the idea that bigebras in general are of fundamental importance in many different areas.

Since in Part II we apply these ideas to the category of C*-algebras, we have to devote Section 6 to some aspects of their structure theory, although basically we take the attitude that the reader is familiar with or willing to check the more elementary facts of this theory in the literature, e.g. in Dixmier's book about this subject [4].

What is needed specifically in our context is a tensor product for C*-algebras, without which the concept of a bigebra could not even be formulated. Even though there is a considerable literature about tensor products of C*-algebras, unfortunately not all questions seem to be clarified. Fortunately for our main applications, we are on excellent ground, since in the case of abelian C*-algebras the tensor product is perfectly understood. Still we discuss the C*-tensor product in somewhat greater generality in Section 7.

This then enables us to go directly from there to the introduction of C*-bigebras and a discussion of basic properties which takes place in Section 8.

In the very short Section 9 we point out the relation between coproducts and tensor products of C*-algebras.

We are then prepared for the duality theory of compact semigroups which now follows effortless and can be recognized as being in many respects a special case of a categorical pattern of great generality. Nevertheless the exploitation and technical details of the present theory are not of a categorical nature but require insight into the special structures at hand: that is understanding of the structure of compact semigroups as of C*-algebras as well. In Section 10 we describe this duality and work out examples of translating properties of compact semigroups into properties of C*-bigebras and vice versa. For instance, the well known fact that any compact semigroup has a unique minimal ideal will appear as a consequence of a much more general theorem about C*-bigebras, namely that every C*-bigebra with identity is colocal, and what this means will be explained in the text. Some quick applications of the duality follow.

The duality between the category of compact semigroups and certain C*-bigebras associates with a compact semigroup again an object which has both an algebraic and a topological structure, although in the case of C*-algebras one has the significant feature that they are "much more algebraic" than other classes of Banach algebras, since topological properties frequently are implied by the algebraic ones. Yet in the spirit of the classical duality theories one would wish to assign a purely algebraic object to a compact semigroup as a dual object. In view of what was said earlier the possible success of such an attempt is questionable; however, in Section 12 we explore to what extent this program is still feasible. In particular we associate with each C*-bigebra an (algebraic) involutive bigebra in a functorial fashion. The original C*-bigebra cannot always be recovered from it, and the extent to which this is possible gives a first indication of the degree of success in finding purely algebraic category as a dual if not for all so at least for a wide class of compact semigroups--including groups. It turns out, perhaps not unexpectedly, that the category of compact semigroups which is well behaved in this respect is the category of semigroups which have sufficiently many linear representations on finite dimensional vector spaces. Such semigroups will be called Peter-Weyl semigroups (for obvious reasons indicating the relationship to the theorem of Peter and Weyl for compact groups).

In Section 13 we derive the classical duality theories from our general one. Following the ideas outlined in HOCHSCHILD's book about Lie groups [8] we add some new facets to shed more light on the question why the existence of Haar measure distinguishes compact groups so significantly from compact semigroups with respect to duality and representation theory. The abstract form of the integral in bigebras has recently been utilized by SWEEDLER [16].

In Section 14 we derive new duality theorems expressing the dual category of the category of totally disconnected compact semigroups with identity in terms of biregular bigebras over the complex numbers. For groups and semilattices one obtains particularly smooth results. These duality theorems in some respect are more algebraic than the general duality in terms of C*-bigebras and are more intrinsic than the general Tannaka duality which involves the somewhat extraneous device of a gauge function.

Section 15, finally, is devoted to the Pontryagin duality of compact and discrete abelian groups in terms of our general duality theory. The first result (15.5), still quite general, shows how the semigroup bigebra of a discrete involutive semigroup fits into our scheme. All groups are involutive relative to inversion, and all abelian semigroups are involutive in a trivial fashion. The variety of objects functorially associated with a discrete involutive semigroup is illustrated in the simple example of the infinite cyclic semigroup, whose dual (in a sense) turns out to be the complex unit disc under multiplication. We discuss the deficiencies of this duality and proceed to show in what respect the situation is drastically better in the case of discrete groups and, perhaps somewhat surprisingly, semilattices. In fact the Pontryagin duality between compact abelian groups and discrete abelian groups has a complete analogue in the duality between compact totally disconnected semilattices with identity and discrete semilattices with identity. We will not, at present, carry out the duality theory, which would cover both of these as special cases, namely the duality between compact abelian Clifford semigroups with identity and totally disconnected semilattice of idempotents and discrete abelian Clifford semigroups with identity which from a somewhat different viewpoint was considered by Schneperman.

A first exposition of the present theory was presented by the author in a course about Compact Groups at Tulane University in 1966/67 while he was a Fellow of the Alfred P. Sloan Foundation. In a sequence of three lectures delivered at the Conference about Topological Semigroups at the University of Florida in Gainesville in April 1969 he reported about some of these developments. Naturally, the topic drew considerable profit from many fruitful discussions with P. S. Mostert. The support by the Alfred P. Sloan Foundation and the National Science Foundation is gratefully acknowledged. Thanks go to Connie Carrier for the care with which she prepared this typescript.

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Section 8. The cofunctor Spec for commutative C*-algebras

Section 9. Coproducts in C*

Section 10. C*-cogebras

Section 11. Duality of compact semigroups

Section 12. Involutive bigebras

Section 13. Dualities for compact groups

Section 14. Dualities for totally disconnected compact semigroups

Section 15. Discrete involutive semigroups and duality