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Olli Tammi

Extremum Problems for Bounded
Univalent Functions



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Preface

This treatise is based on my seminar lectures and resulting discussions in Helsinki University during the academic years 1974-1977. It is my aim to give a survey of the main methods concerning univalent functions, i.e. the use of Löwner's functions, Schiffer's differential equations and Grunsky type inequalities. These shall be tested by deriving certain coefficient results. It appears especially, that some basic information concerning coefficient bodies can be found from defined Grunsky type conditions. On the other hand, Schiffer's differential equation characterizes all boundary functions of the coefficient body. Hence, a natural problem in the field of inequalities arises: The Grunsky type inequalities must be so extended that they are sharp for these Schiffer-functions. One possible method here is the use of Löwner's functions. This survey is an introduction to that task.

Only those works that are directly related to the theme are given in the Reference list.

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Helsinki, March 1977

Olli Tammi

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