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A. Frölicher · W. Bucher
Université de Genève

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INTRODUCTION

As emphasized by J. Dieudonné, calculus primarily deals with the approximation (in a neighborhood of some point) of given mappings of vector spaces by linear mappings. The approximating linear map has to be a "good" approximation in some precise sense: it has to be "tangent" to the given map. A very useful notion of "tangent" can easily be introduced for maps between normed vector spaces; it leads to the notion of "Fréchet-differentiable" mappings and gives, in particular for Banach spaces, a very satisfactory theory (cf. Chap. VIII of [3]).

It is well known that in this classical theory the notions of differentiability and derivative remain unchanged if one replaces the given norms by equivalent ones, i.e. by norms inducing the same topologies. It is natural therefore to look for a theory which does not use the norms, but only the topologies of the considered vector spaces. In fact, throwing out something which is irrelevant usually leads to a clarification and simplification on one side, and allows a more general theory on the other side. In the case of calculus, such a generalization is indeed desirable in view of applications to certain function spaces which have a natural topology, but no natural norm.

In classical theory, the norm is essentially used at two places: (1) One defines what maps $r: E_1 \longrightarrow E_2$ are tangent to zero at the origin (we simply call them "remainders") by means of the Fréchet-condition: $\lim_{x \to 0} \left(\frac{1}{\|x\|} \cdot \|r(x)\|\right) = 0$: (2) One defines a norm on the vector space $L(E_1;E_2)$, consisting of the continuous linear maps from E_1 into E_2, by taking, for $\ell \in L(E_1;E_2)$: $\|\ell\| = \sup_{\|x\| = 1} \|\ell(x)\|$. In order to obtain a similar theory for a class of non-normed topological vector spaces, one has therefore to choose essentially two definitions: (1) What are the remainders from E_1 to E_2;

(2) What is the topology of $L(E_1;E_2)$. The second definition comes in as soon as one wants to consider second (or higher) derivatives, since the first derivative f' of a (differentiable) map $f: E_1 \longrightarrow E_2$ is a map $f': E_1 \longrightarrow L(E_1;E_2)$. But all attempts which have been made along this line gave theories with a very serious deficiency: the composite of twice differentiable mappings did not turn out to be twice differentiable in general; in other words: there was no higher order chain rule. In fact, a look at the classical proof shows that the second order chain rule is a consequence of the first order chain rule and of the differentiability of the composition map $c: L(E_1;E_2) \times L(E_2;E_3) \longrightarrow L(E_1;E_3)$. But for non-normable topological vector spaces E_i there seems to be no separated topology on the spaces $L(E_i;E_j)$ such that the composition becomes differentiable.(*) Nevertheless, a way out of this difficulty was found: independently A. Bastiani and H.H. Keller realized that though there is no satisfactory topology on the spaces $L(E_i;E_j)$, there exist pseudo-topologies which have the desired properties. The authors are very much indebted to H.H. Keller for having drawn their attention to the fact that pseudo-topologies seem really the proper thing to use at this place.

(*) This statement is not very precise, in particular since it depends on the adopted definition of "differentiable". If, however, one requires that "differentiable" shall imply "continuous" and that the natural isomorphism between $L(\mathbf{R};E_i)$ and E_i shall be a homeomorphism, then one knows that with topologies one cannot succeed; in fact, the continuity of the composition map $c: L(\mathbf{R};E_1) \times L(E_1;E_2) \longrightarrow L(\mathbf{R};E_2)$ then is equivalent with the continuity of the evaluation map $e: L(E_1;E_2) \times E_1 \longrightarrow E_2$, and for non-normable spaces E_1, E_2 there is no topology on $L(E_1;E_2)$ for which this evaluation map e is continuous (cf. [7]).

The above remarks show that it is not for the sake of greatest possible generality that we develop our theory right from the beginning for pseudo-topological vector spaces (topological ones are special cases of these), but simply in order to obtain a satisfactory theory for a class of vector spaces containing at least some non-normable topological ones. In order to prove certain theorems of calculus, some restrictions however will have to be made: a class of pseudo-topological vector spaces, called "admissible" ones, will be introduced. This class contains in particular all separated locally convex topological vector spaces.

Since our whole theory works consistently with filters, §1 starts with some well known facts concerning filters. For a reader who is familiar with filters, it will be sufficient to have a look at (1.5.2); we found that at some places in the literature the inequality stated there was erroneously used as an equality. §2 presents the basic facts concerning pseudo-topologies and in particular pseudo-topological vector spaces. The material of sections 2.5 to 2.9 will not be used for the beginning of calculus and thus can be read later, whenever referred to.

§5 deals with what might be called the "mean value theorem". However, there is no mean value in it; but it is fundamental in the sense that it is used in order to prove practically all of the deeper results of calculus. We thus call it "fundamental theorem of calculus". Intuitively, it gives an estimate of the difference between the endpoints of a motion of a point in a vector space by means of the velocity of that motion, the estimation being made by means of convex sets. In the case of normed spaces, the theorem yields the well known estimate by means of the norm (cf. (8.5.1) of [3]) provided one chooses as convex set the closed unit ball; but being able to take other convex sets, we get better information also in this classical case: we not only can conclude that the point does not get too far if the velocity is not too big, but also that the point does get far, if the velocity is big (in the sense of lying in a multiple of the convex set in question). For later applications, some consequences of the theorem are established at the end of §5; in particular, two versions of the theorem in the form of filter inequalities will turn out to be useful. Another consequence is Taylor's formula; it will be given in a forthcoming publication.

In §7 one finds the definition of the admissible spaces and furthermore a result without which the theory would not be satisfactory: the class of admissible vector spaces is closed under the constructions used in calculus, yielding new spaces out of given ones, such as $L(E_1;E_2)$ or $C_k(E_1;E_2)$ out of E_1 and E_2.

In §8 we show that the relations between partial and total differentiability of a mapping of a direct product are as in classical theory; in particular, partial differentiability plus continuity of the partial derivatives implies total differentiability. We remark that this theorem uses in a very essential way the choice of the structure of the spaces $L(E_i;E_j)$, since "continuous" refers to the pseudo-topologies of the spaces in question.

The main results of §9 state that the p-th derivative at a point can be identified with a multilinear map which is symmetric, and that the composite of p-times differentiable maps is also p-times differentiable (p-th order chain rule).

The notion of a C_k-mapping introduced in §10 coincides with the usual notion of a k-times continuously differentiable mapping in the case of finite-dimensional spaces, while in general it is slightly more restrictive. The vector space consisting of the C_k-mappings from E_1 into E_2 is denoted by $C_k(E_1;E_2)$ or $C_k^*(E_1;E_2)$, depending on which of two pseudo-topological structures we consider (we always use one symbol to denote the space and its structure). The important spaces are the spaces $C_k^{*}(E_1;E_2)$; but for technical reasons it is useful to define them by means of the spaces $C_k(E_1;E_2)$ as auxiliary spaces and a general operator " $*$ " which associates to any pseudo-topology of a vector space a second one, having in addition a certain important property, called equability. In special cases, the operator " $*$ " becomes the identity; in particular, if the spaces E_i are finite dimensional, the pseudo-topology of $C_k(E_1;E_2) = C_k^{*}(E_1;E_2)$ is nothing else than the topology of uniform convergence on bounded sets of the functions and their derivatives up to the k-th order. The case $k = \infty$ is obtained by forming a projective limit.

In §11 the differentiability and the C_p-nature of the composition map of C_k-mappings are investigated, the main results being theorems (11.2.21) and (11.2.26); here, the result stating that the composition map is of class C_p is in fact stronger than just saying that it is p-times continuously differentiable.

§12 deals with differentiable families of differentiable maps, "differentiable" now always meaning "differentiable of class C_∞". Having our theory of differentiation and also a pseudo-topology on the vector space of differentiable maps from E_1 into E_2, one can consider two sorts of differentiable families of such maps: a) A differentiable family of maps (depending, for instance, on a real parameter) is a differentiable map of $\mathbb{R} \times E_1$ into E_2; b) A differentiable family of maps is a differentiable map of $\mathbb{R}$ into the function space $C_\infty^{*}(E_1;E_2)$. The main result of §12 not only says that these two notions are completely equivalent, but even that the structures put on the space of all differentiable families according to either one of the two points of view a) or b) are the same; in other words, there is a canonical linear homeomorphism between $C_\infty^{*}(\mathbb{R} \times E_1;E_2)$ and $C_\infty^{*}(\mathbb{R};C_\infty^{*}(E_1;E_2))$. Moreover, the "parameter space" $\mathbb{R}$ can be replaced by any admissible equable vector space E. If we consider this isomorphism in the special case $E = E_1 = E_2 = \mathbb{R}$, for instance, then the space on the left hand side is classically well known, while on the right hand side we have a new function space, consisting of functions with values in the infinite dimensional space $C_\infty(\mathbb{R};\mathbb{R})$. Repeating this argument one sees that at least for many spaces E_i, E_j the set and the structure of $C^{\#}_\infty(E_i;E_j)$ are uniquely determined if one requires the following two conditions: (1) in case E_i and E_j are finite dimensional, $C^{\#}_\infty(E_i;E_j)$ is the set of classical C_∞-functions, with the topology of uniform convergence on compact sets of the functions and their derivatives; (2) the linear homeomorphism (12.2.5) mentioned before shall hold.

Depending on the choice of the two main definitions one obtains different theories. Our approach is different from those of A. Bastiani, H.H. Keller and E. Binz ([1],[6],[2]). In order to develop our theory, we always postulated that the definitions agree with the classical ones in the case of normed spaces, a condition which is not satisfied by the theories of A. Bastiani or of E. Binz. The structure of $L(E_1;E_2)$ defined by H.H. Keller for the case of locally convex spaces E_1, E_2 by means of families of semi-norms (cf. [5]), seems to be the same as the structure of our $L^{\#}(E_1;E_2)$. In [6], H.H. Keller introduces various notions of differentiability and compares them with definitions that have been suggested by still other authors (cf. the references given there), restricting himself in that paper to locally convex spaces.

At the time being it is difficult to recognize which one of the various theories will eventually turn out to be the most useful one. That mainly depends on what theorems one gets and on what applications one wants to make. An implicit function theorem has not been obtained so far; in fact it is known that its classical formulation simply fails to hold. H.H. Keller has also established and motivated a series of basic properties that should hold in a useful theory of calculus (cf. [7]); we believe that our theory satisfies these conditions.

Throughout this report, we restrict ourselves to certain vector spaces; manifolds modelled on such vector spaces shall be considered later.

Though our notion of differentiability is a local property, a non-local condition is imposed on the so-called C_k-mappings; but this condition becomes trivial in the case of finite dimensional spaces, and, at least, it is not so restrictive as to rule out the identity map, as it would be the case if one had to restrict oneself to maps with compact or bounded support.

The first-named author has presented a first version of calculus for topological vector spaces in a Seminar of Professors A. Dold and B. Eckmann at the Swiss Federal Institute of Technology (ETH), Zurich, in summer 1963; it was not yet satisfactory, since there was no higher order chain rule. A part of the present theory was outlined by the same author in a series of lectures at the Forschungsinstitut für Mathematik of the ETH during the 1964/65 winter term.

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