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Pierre Collet   Jean-Pierre Eckmann

Concepts and Results in Chaotic Dynamics: A Short Course

With 67 Figures

Springer

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To Sonia and Doris

Preface

This book is devoted to the subject commonly called Chaotic Dynamics, namely the study of complicated behavior in time of maps and flows, called dynamical systems.

The theory of chaotic dynamics has a deep impact on our understanding of Nature, and we sketch here our view on this question. The strength of this theory comes from its generality, in that it is not limited to a particular equation or scientific domain. It should be viewed as a conceptual framework with which one can capture properties of systems with complicated behavior. Obviously, such a general framework cannot describe a system down to its most intricate details, but it is a useful and important guideline on how a certain kind of complex systems may be understood and analyzed.

The theory is based on a description of idealized systems, such as "hyperbolic" systems. The systems to which the theory applies should be similar to these idealized systems. They should correspond to a *fixed* evolution equation, which, however, need to be neither modeled nor explicitly known in detail. Experimentally, this means that the conditions under which the experiment is performed should be as constant as possible. The same condition applies to analysis of data, which, say, come from the evolution of glaciations: One cannot apply "chaos theory" to systems under varying external conditions, but only to systems which have some self-generated chaos under fixed external conditions.

So, what *does* the theory allow us to do? We can measure indicators of chaos, and study their dependence on those fixed external conditions. Is the system's behavior regular or chaotic? This can be, for example, inferred by measuring Lyapunov exponents. In general, the theory tells us that complex systems should be analyzed statistically, and not, as was mostly done before the 1960s, by all sorts of Fourier-mode- and linearized, analysis. We hope that the present book and in particular Sect. 9 shows what the useful and robust indicators are.

The material of this book is based on courses we have given. Our aim is to give the reader an overview of results which seem important to us, and which are here to stay. This book is not a mathematical treatise, but a course, which tries to combine two slightly contradicting aims: On one hand to present the main ideas in a simple way and to support them with many examples; on the other to be mathematically sufficiently precise, without undue detail. Thus, we do not aim to present the most general results on a given subject, but rather explain its ideas with a simple statement and many examples. A typical instance of this restriction is that we tacitly assume enough regularity to allow for a simpler exposition.

The proofs of the main results are often only sketched, because we believe that it is more important to understand how the concepts fit together in leading to the results than to present the full details. Thus, we usually spend more space on explaining the ideas than for the proofs themselves. This point of view should enable the reader to grasp the essence of a large body of ideas, without getting lost in technicalities. For the same reason, the examples are carefully chosen so that the general ideas can be understood in a nutshell.

The level of the book is aimed at graduate students in theoretical physics and in mathematics. Our presentation requires a certain familiarity with the language of mathematics but should be otherwise mostly self-contained.

The reader who looks for a mathematical treatise which is both detailed and quite complete, may look at (de Melo and van Strien 1993; Katok and Hasselblatt 1995). For the reader who looks for more details on the physics aspects of the subject a large body of literature is available, with different degrees of mathematical rigor: (Eckmann 1981) and (Eckmann and Ruelle 1985a) deal with experiments of the early 1980s; (Manneville 1990; 2004) deals with many experimental setups; (Abarbanel 1996) is a short course for physicists; (Peinke, Parisi, Rössler, and Stoop 1992) concentrates on semiconductor experiments; (Golubitsky and Stewart 2002) has a good mix of mathematical and experimental examples. Finally, (Kantz and Schreiber 2004) deal with nonlinear time series analysis.

The references in the text cover many (but obviously not all) original papers, as well as work which goes much beyond what we explain. In this way, the reader may use the references as a guide for further study. The reader interested in more of an overview will also find references to textbooks and monographs which shed light on our subject either from different angle, or in the way of more complete treatises.

Like any such project, to remain of reasonable size, we have omitted several subjects which might have been of interest; in particular, bifurcation theory (Arnold 1978; Ruelle 1989b; Guckenheimer and Holmes 1990), topological dynamics, complex dynamics, the Kolmogorov–Arnold–Moser (KAM) theorem and many others. In particular, we mostly avoid repeating material from our earlier book (Collet and Eckmann 1980).

After a few introductory chapters on dynamics, we concentrate on two main subjects, namely hyperbolicity and its consequences, and statistical properties of interest to measurements in real systems.

To make the book easier to read, several definitions and some examples are repeated, so that the reader is not obliged to go back and forth too often.

We hope that the many illustrations, simple examples, and exercises help the reader to penetrate to the core of a beautiful and varied subject, which brings together ideas and results developed by mathematicians and physicists.

This book has profited from the questions, suggestions and reactions of the numerous students in our courses: We warmly thank them all. Furthermore, our work was financially supported by the Fonds National Suisse, the European Science Foundation, and of course our host institutions. We are grateful for this support.

Paris and Geneva,
May 2006

Pierre Collet
Jean-Pierre Eckmann

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