Part IV

The Incompressible Navier-Stokes Equations

The unsteady incompressible Navier-Stokes equations,

$$\begin{aligned} \mathbf{u}_t - \nu \triangle \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} &\quad \text{in } \Omega \times (0,T], \\ \nabla \cdot \mathbf{u} = 0 &\quad \text{in } \Omega \times (0,T], \\ \mathbf{u} = \mathbf{u}_b &\quad \text{on } \partial\Omega \times (0,T], \\ \mathbf{u}(0) = \mathbf{u}_0 &\quad \text{in } \Omega, \end{aligned}$$

are widely studied as a valuable model in the highly significant research area of Computational Fluid Dynamics (CFD); see, e.g., [Fei93, FŠ04, KL04, Gal94, GR86, GS00a, GS00b, Gun89, Tem83]. In these equations $\nu := 1/\mathrm{Re}$ is the inverse of the Reynolds number, $\mathbf{u} = (u_1, \ldots, u_d)$ is the unknown velocity, p is the pressure field, $\mathbf{f} = (f_1, \ldots, f_d)$ a given body force, $\mathbf{u}_b$ a prescribed velocity field at the boundary, $\mathbf{u}_0$ the velocity field at time $t = 0$, Ω a bounded domain in $\mathbb{R}^d$ (where $d = 2$ or 3) with Lipschitz-continuous boundary, and $(0, T]$ the time interval considered.

Throughout Part IV boldface letters will be used (as above) to denote vector-valued quantities. For notational convenience when discussing spaces like $L_0^2(\Omega)$ we write L^2 instead of L_2, unlike Parts I–III. Furthermore, we sometimes use the notation $\|\cdot\|_{m,p}$ and $|\cdot|_{m,p}$ for the norm and highest-order seminorm in the Sobolev space $W^{m,p}(\Omega)$; thus for example $|\cdot|_{1,2} \equiv |\cdot|_1$ and $\|\cdot\|_{0,2} \equiv \|\cdot\|_0$.

Written out in full, these differential equations are

$$\frac{\partial u_i}{\partial t} - \nu \Delta u_i + \sum_{j=1}^{d} u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} = f_i \quad \text{in } \Omega \times (0,T], \quad \text{for } i = 1, \ldots, d,$$

$$\sum_{i=1}^{d} \frac{\partial u_i}{\partial x_i} = 0 \quad \text{in } \Omega \times (0,T].$$

The first d equations here model conservation of momentum while the final equation states that mass is conserved.

Implicit time discretizations of the Navier-Stokes equations lead at each time step to the Oseen problem

$$
\begin{aligned}
-\nu\triangle\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{u} + \sigma\mathbf{u} + \nabla p = \tilde{\mathbf{f}} \quad & \text{in } \Omega, \\
\nabla\cdot\mathbf{u} = 0 \quad & \text{in } \Omega, \\
\mathbf{u} = \tilde{\mathbf{u}}_b \quad & \text{on } \partial\Omega,
\end{aligned}
$$

where $\tilde{\mathbf{f}}$ and $\tilde{\mathbf{u}}_b$ depend on the solution at the previous time step, $\mathbf{b}$ is a given vector field with $\nabla\cdot\mathbf{b} = 0$, and $\sigma \sim 1/\triangle t$. The Oseen problem also arises when solving the nonlinear stationary Navier-Stokes equations

$$
\begin{aligned}
-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad & \text{in } \Omega, \\
\nabla\cdot\mathbf{u} = 0 \quad & \text{in } \Omega, \\
\mathbf{u} = \mathbf{u}_b \quad & \text{on } \partial\Omega
\end{aligned}
$$

by a fixed-point iteration. In this case $\mathbf{b}$ corresponds to the previous iterate $\mathbf{u}^{\text{old}}$ of the velocity field and $\sigma = 0$.

Part IV will show how methods developed for convection-diffusion problems can be applied to this more complex system of equations. Compared with Parts I–III, there are additional difficulties in the numerical solution of the Oseen and Navier-Stokes problems:

- in two space dimensions, the Navier-Stokes equations with Dirichlet boundary data have, on any time interval $[0, T]$, a unique solution that is also a classical solution provided that all data of the problem are smooth enough; but in three dimensions, the existence of such solutions has been proved only for sufficiently small data or on sufficiently short intervals of time.
- to prove uniqueness of any solution of the stationary version of the Navier-Stokes equation, a smallness restriction on the data such as (1.5) below is needed in both two and three dimensions; uniqueness cannot be guaranteed for all positive ν and all data $\mathbf{f}$ and $\mathbf{u}_b$.
- the incompressibility constraint $\nabla\cdot\mathbf{u} = 0$ does not, in general, allow arbitrary approximations of the velocity and pressure fields – the approximation spaces must satisfy an inf-sup condition (the Babuška-Brezzi condition) or an additional "pressure stabilization" (like those considered in Chapters 3 and 4) has to be introduced.
- the nonlinear convection term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ couples different components u_i of the solution.
- the Newton linearization of the momentum equation fails in general to be coercive; moreover, the dependence on ν of the norm of its inverse is *a priori* unknown; see Remark 1.3 below for more details.

These complications make the analysis of numerical methods for the Navier-Stokes equations a formidable task. In particular, some familiar and useful theoretical tools – e.g., the maximum principle – cannot be used. Discretizations of the incompressible Navier-Stokes problem by finite element methods suffer in general from two main shortcomings. First, the discrete inf-sup (Babuška-Brezzi) condition is violated. Second, spurious oscillations occur because of the predominantly convective nature of the equations. Both these shortcomings are present also in the Oseen problem which is linear and uniquely solvable for all positive ν and all data $\mathbf{f}$ and $\mathbf{u}_b$. It is thus unsurprising that in the research literature, the Oseen problem is seen as a suitable test bed for the development of robust and efficient numerical methods for the incompressible Navier-Stokes equations.

Our investigation in Part IV will confine itself mainly to the Oseen equation (linear) and the stationary Navier-Stokes equations (nonlinear) with the homogeneous Dirichlet boundary condition $\mathbf{u}_b = \mathbf{0}$. For inhomogeneous boundary conditions, see [GP83, Gun96]. As regards the various stabilization techniques of Part III, we shall restrict ourselves to upwind finite element methods, methods of streamline diffusion (SDFEM) type, and local projection stabilization (LPS) methods. For the application of the continuous interior penalty (CIP) method we refer to [BFH06, BH06, Bur07] and the discontinuous Galerkin (dGFEM) approach is considered in [CKS04, CKS05b, CKS05a, CKS07].

For the unsteady Navier-Stokes equations, standard finite element methods are analysed in [HR82, HR86, HR88, HR90] and the survey papers [Ran94, Ran00, Ran04] also discuss stability issues. Applications of the SDFEM in a space-time setting are investigated in [JS86] and [HS90]. The semi-discretization of the unsteady Navier-Stokes equations using the CIP approach is studied in [BF07].