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Robust Numerical Methods for Singularly Perturbed Differential Equations

Convection-Diffusion-Reaction and Flow Problems

Second Edition
With 41 Figures

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Preface

The analysis of singular perturbed differential equations began early in the twentieth century, when approximate solutions were constructed from asymptotic expansions. (Preliminary attempts appear in the nineteenth century – see [vD94].) This technique has flourished since the mid-1960s and its principal ideas and methods are described in several textbooks; nevertheless, asymptotic expansions may be impossible to construct or may fail to simplify the given problem and then *numerical approximations* are often the only option.

The systematic study of numerical methods for singular perturbation problems started somewhat later – in the 1970s. From this time onwards the research frontier has steadily expanded, but the exposition of new developments in the analysis of these numerical methods has not received its due attention. The first textbook that concentrated on this analysis was [DMS80], which collected various results for ordinary differential equations. But after 1980 no further textbook appeared until 1996, when *three* books were published: Miller et al. [MOS96], which specializes in upwind finite difference methods on Shishkin meshes, Morton's book [Mor96], which is a general introduction to numerical methods for convection-diffusion problems with an emphasis on the cell-vertex finite volume method, and [RST96], the first edition of the present book. Nevertheless many methods and techniques that are important today, especially for partial differential equations, were developed after 1996. To give some examples, layer-adapted special meshes are frequently used, new stabilization techniques (such as discontinuous Galerkin methods and local subspace projections) are prominent, and there is a growing interest in the use of adaptive methods. Consequently contemporary researchers must comb the literature to gain an overview of current developments in this active area. In this second edition we retain the exposition of basic material that underpinned the first edition while extending its coverage to significant new numerical methods for singularly perturbed differential equations.

Our purposes in writing this introductory book are twofold. First, we present a structured and comprehensive account of current ideas in the numerical analysis of singularly perturbed differential equations. Second, this important area has many open problems and we hope that our book will stimulate their investigation. Our choice of topics is inevitably personal and reflects our own main interests.

We have learned a great deal about singularly perturbed problems from other researchers. We thank those colleagues who helped and influenced us; these include V.B. Andreev, A.E. Berger, P.A. Farrell, A. Felgenhauer, E.C. Gartland, Ch. Großmann, A.F. Hegarty, V. John, R.B. Kellogg, N. Kopteva, G. Lube, N. Madden, G. Matthies, J.J.H. Miller, K.W. Morton, F. Schieweck, G.I. Shishkin, E. Süli, and R. Vulanović; in particular Herbert Goering and Eugene O'Riordan guided our initial steps in the area. Our research colleague T. Linß deserves additional thanks for providing many of the figures in this book.

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Part II Parabolic Initial-Boundary Value Problems in One Space Dimension

Part IV The Incompressible Navier-Stokes Equations

Notation

I	identity
L	differential operator
L^*	adjoint operator
$a(\cdot,\cdot)$	bilinear form
g, G	Green's function
V, V^*	Banach space and the corresponding dual space
V_h	finite-dimensional subspace of V
$\|\|\cdot\|\|_V$	norm on the space V
$\|\|\cdot\|\|_{*,d}$	discrete version of the norm $\|\|\cdot\|\|_*$
$r \cdot s$	scalar product of vectors in $\mathbb{R}^d$
$(\cdot,\cdot)$	scalar product in Hilbert space
$f(v)$ or $\langle f, v\rangle$	functional f applied to v
$\|\|f\|\|_*$	norm of the functional f
$U \hookrightarrow V$	continuous embedding of U in V
Ω	given space variable(s) domain
$\partial\Omega = \Gamma$	boundary of Ω
$meas\,(\Omega)$	measure of Ω
n	outward-pointing unit vector normal to $\partial\Omega$
t, T	time with $t \in (0,T)$
$Q = \Omega \times (0,T)$	given domain for nonstationary problems
$C^l(\Omega)$, $C^{l,\alpha}(\Omega)$	function spaces
$L_p(\Omega)$	function space, $1 \le p \le \infty$
$\|\|\cdot\|\|_{0,p}$	norm in $L_p(\Omega)$
$\|\|\cdot\|\|_{L_p,d}$	discrete norm in $L_p(\Omega)$
$W^{m,p}(\Omega), \\|\cdot\\|_{m,p,\Omega}$	Sobolev spaces and their norms
$H^l(\Omega), H^l_0(\Omega)$	Sobolev spaces $W^{1,2}(\Omega)$
$\|\|\cdot\|\|_l, \|\cdot\|_l$	norm and seminorm in $H^l(\Omega)$
$\|\|\cdot\|\|_{l,E}$	H^l-norm restricted to $E \subset \Omega$
ε	singular perturbation parameter
C	generic constant, independent of ε

$\|\| \cdot \|\|_\varepsilon$	ε-weighted $H^1(\Omega)$ norm
$\|\| \cdot \|\|_{gr}$	graph norm
∇ or *grad*	gradient
div, div $c = \nabla \cdot c$	divergence operator
$\mathcal{O}(\cdot),\ o(\cdot)$	Landau symbols
P_r	polynomials of degree at most r
P_r^{disc}	piecewise polynomials of degree at most r, discontinuous across element boundaries
Q_r	products of polynomials of degree at most r
Q_r^{disc}	products of polynomials of degree at most r, discontinuous across element boundaries
h, h_i	mesh parameter in space
τ, τ_j	mesh parameter in time
L_h	difference operator
D^+, D^-, D^0	difference quotients
$\triangle, \triangle_h$	Laplacian and its discretization
$\omega_h,\ \Omega_h$	set of meshpoints
$u, u_h, u_i, u_i^j, u_{ij}$	unknown(s)
u_0	reduced solution
I_h	interpolation operator
$u^I = I_h\, u$	nodal interpolant of u
$\pi_h u, \Pi_h u, i_h u$	quasi-interpolant of u, defined for non-smooth functions u

mesh-dependent norms are written with three vertical lines: $\|\|\| \cdot \|\|\|$

$\|\|\| \cdot \|\|\|_{SD}$	norm used in streamline diffusion finite element method
$\|\|\| \cdot \|\|\|_{CIP}$	norm used in continuous interior penalty finite element method
$\|\|\| \cdot \|\|\|_{LPS}$	norm used in local projection stabilization finite element method
$\|\|\| \cdot \|\|\|_{dG}$	norm used in discontinuous Galerkin finite element method
$\|\|\| \cdot \|\|\|_{GLS}$	norm used in the Galerkin least-squares finite element method

Introduction

Imagine a river – a river flowing strongly and smoothly. Liquid pollution pours into the water at a certain point. What shape does the pollution stain form on the surface of the river?

Two physical processes operate here: the pollution *diffuses* slowly through the water, but the dominant mechanism is the swift movement of the river, which rapidly *convects* the pollution downstream. Convection alone would carry the pollution along a one-dimensional curve on the surface; diffusion gradually spreads that curve, resulting in a long thin curved wedge shape.

When convection and diffusion are both present in a linear differential equation and convection dominates, we have a *convection-diffusion problem.*

The simplest mathematical model of a convection-diffusion problem is a two-point boundary value problem of the form

$$-\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x) \quad \text{for } 0 < x < 1,$$

with $u(0) = u(1) = 0$, where ε is a small positive parameter and a, b and f are some given functions. Here the term u'' corresponds to diffusion and its coefficient $-\varepsilon$ is small. The term u' represents convection, while u and f play the rôles of a source and driving term respectively. (Spriet and Vansteenkiste [SV82] explain why diffusion and convection should be modelled by second-order and first-order derivatives respectively.)

Example 0.1. Consider the problem

$$-\varepsilon u''(x) + u'(x) = 1 \qquad \text{for} \quad 0 < x < 1, \tag{0.1}$$

with $u(0) = u(1) = 0$ and $0 < \varepsilon \ll 1$.

Suppose that we set formally $\varepsilon = 0$ here. This yields

$$u'(x) = 1 \qquad \text{for} \quad 0 < x < 1, \tag{0.2}$$

with $u(0) = u(1) = 0$. Unlike (0.1) this problem has no solution in $C^1[0, 1]$. We infer that when ε is near zero, the solution of (0.1) is badly behaved in some way. ♣

Problems like (0.1) form the subject matter of this book. They are differential equations (ordinary or partial) that depend on a small positive parameter ε and whose solutions (or their derivatives) approach a discontinuous limit as ε approaches zero. Such problems are said to be *singularly perturbed*, where we regard ε as a perturbation parameter. In more technical terms, one cannot represent the solution of a singularly perturbed differential equation as an asymptotic expansion in powers of ε.

The solutions of singular perturbation problems typically contain *layers*. Ludwig Prandtl introduced the terminology *boundary layer* at the Third International Congress of Mathematicians in Heidelberg in 1904. (Prandtl's paper, "Über Flüssigkeitsbewegung bei sehr kleiner Reibung", is one of the most influential applied mathematics papers of the 20th century.) To see how such layers arise, consider the following time-dependent Navier-Stokes problem in two space variables x and y:

$$\frac{\partial u}{\partial t} - \frac{1}{\mathrm{Re}}\triangle u + (u \cdot \nabla)u = -\nabla p \quad \text{in the upper half-plane } y > 0, \tag{0.3a}$$

$$\nabla \cdot u = 0 \quad \text{in the same domain,} \tag{0.3b}$$

$$u = 0 \quad \text{on the boundary } y = 0, \tag{0.3c}$$

at large Reynolds number Re. One can regard the boundary $y = 0$ as a fixed plate, and we assume that the velocity u at $y = \infty$ is parallel to the x-axis with magnitude U. We seek a flow, at constant pressure p, whose velocity is parallel to the plate and independent of x. Then equation (0.3a) reduces to

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial y^2}, \quad \text{where} \quad \varepsilon = \frac{1}{\mathrm{Re}}.$$

Set $\eta = y/(2\sqrt{\varepsilon t})$ and let $u(y,t) = U\, f(\eta)$. A computation leads to

$$u = 2U \operatorname{erf}(\eta), \quad \text{where} \quad \operatorname{erf}(\eta) = \frac{1}{\sqrt{\pi}} \int_0^\eta e^{-s^2} ds. \tag{0.4}$$

Equation (0.4) shows that there is a narrow region near $y = 0$ where u departs significantly from the constant flow U. We say that u has a *boundary layer* at $y = 0$. See [CM93] for a detailed discussion. Linearization of (0.3) yields an equation of the form

$$\frac{\partial u}{\partial t} - \varepsilon \triangle u + b \cdot \nabla u + cu = f,$$

where b is independent of u. Such convection-diffusion equations model many fluid flows [Hir88, KL04]; they appear in the well-known Oseen equations and in related subjects like water pollution problems [REI$^+$07], simulation of oil extraction from underground reservoirs [Ewi83], flows in chemical reactors [Alh07] and convective heat transport problems with large Péclet numbers [Jak59].

Of course, convection-diffusion equations do not arise only in fluid flows; the next illustration comes from semiconductor device simulation.

Example 0.2. The "continuity equation" for electrons [PHSM87] in a steady-state scaled model of a one-dimensional semiconductor – with several simplifying assumptions – is

$$\frac{d^2n}{dx^2} - \frac{d}{dx}\left[n\frac{d}{dx}(\psi + \log n)\right] = 0, \tag{0.5}$$

where the unknown function n is the electron concentration, and ψ (which is computed from another part of the model) is the electrostatic potential. Now $d\psi/dx$ is typically very large (perhaps 10^5) on part of its domain (see [PHSM87, Figure 2]), so the unit coefficient of the diffusion term d^2n/dx^2 will be dominated there by the convection term coefficient. That is, equation (0.5) is a convection-diffusion problem. ♣

Singularly perturbed differential equations appear in several branches of applied mathematics. (We have seen only two examples, albeit significant ones.) The analysis and numerical solution of convection-diffusion problems deservedly attracts substantial attention.

In this book, we discuss the nature of solutions of various singularly perturbed differential equations before presenting methods for their numerical solution. Thus Part I begins with an exposition of the technique of matched asymptotic expansions, which is then used to examine various classes of two-point boundary value problems. In Part II we move on to time-dependent problems with one space dimension. Elliptic and parabolic problems in several space dimensions come in Part III. Finally, Part IV discusses finite element methods for a significant applied model: the Navier-Stokes equations.

If any discretization technique is applied to a parameter-dependent problem, then the behaviour of the discretization depends on the parameter. For singularly perturbed problems, conventional techniques often lead to discretizations that are worthless if the singular perturbation parameter is close to some critical value. We are interested in *robust* methods that work for all values of the singular perturbation parameter. We therefore track carefully the dependence on this parameter of those constants that arise in consistency, stability and error estimates. Thus the philosophy of this book emphasizes *realistic error estimates.* This contrasts sharply with much published research whose analysis ignores the effect of parameter dependence. There is a growing awareness of the dangers of this neglect; in the particular case of the incompressible Navier-Stokes equations, Johnson, Rannacher and Boman [JRB95a] observe that existing analyses often contain constants that depend on exp(Re), where Re is the Reynolds number, and conclude that "in the majority of cases of interest, the existing error analysis has no meaning". We hope that the careful approach that is followed here will provide a serviceable foundation for future work.

Discretization leads to a linear or nonlinear system of equations with a large number of unknowns. Iterative methods are commonly used to solve these systems. It is important to realize that these solvers, like the underlying discretization, should be robust with respect to the singular perturbation parameter. The discretization of a convection-diffusion problem usually produces a nonsymmetric system of equations and this asymmetry complicates the linear algebra analysis. No attempt is made in this book to discuss these issues; instead the recent textbook of Elman, Silvester and Wathen [ESW05] is recommended.

In general standard notation is used for function spaces, norms, etc. (see the notation list on page XIII), but two special conventions should be noted. First, the unknown u in a singular perturbation problem depends, of course, on the perturbation parameter ε. While one must always bear this dependence in mind, it is not included in our notation; that is, we write $u(x)$ instead of, for instance, $u(x,\varepsilon)$ or $u_\varepsilon(x)$. This simplifies the notation, especially when the discretization requires the use of some indices that depend on the mesh. On the other hand, an expression like $\lim_{\varepsilon\to 0} u(x)$ then looks odd, but one should remember that the unknown u does depend on ε. Every notation has its advantages and disadvantages! Second, in our analysis it is important to declare whether or not each constant depends on ε. Thus we denote by C (sometimes subscripted or superscripted) a *generic constant* that is always *independent of the perturbation parameter and of any mesh used.* Other letters are used to denote other "constants" when such a dependence is present.

The following example illustrates our system of numbering and internal cross-referencing. In Part I, Theorem 1.4 lies in Chapter 1 (hence the numbering "1.$*$"). In Part I it is referred to as "Theorem 1.4", but we call it "Theorem I.1.4" when it's referred to from outside Part I. A similar convention is used for equations, Lemmas, etc.

We assume that the reader is familiar with the basic theory of ordinary and partial differential equations, and with the jargon and usage of finite difference and finite element methods.

Finally, despite our best efforts, mistakes are undoubtedly present in this book. We invite each reader to email us [rst-book@ovgu.de] any corrections that s/he notices, and this information will be made publicly available at the website [www.rst-book.ovgu.de].