Scientific Computation

V. D. Liseikin

A Computational Differential Geometry Approach to Grid Generation

Second Edition

With 81 Figures
Including 3 Color Figures

Springer

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Preface to the Second Edition

This second edition of *A Computational Differential Geometry Approach to Grid Generation* is significantly expanded by new material that centers on the recent advances in grid generation technology based on the numerical solution of Beltrami and diffusion equations in monitor metrics. It gives a more detailed and practice-oriented description of the monitor metrics for providing the generation of adaptive, field-aligned, and balanced numerical grids. New finite-difference codes are described for generating both structured and unstructured surface and domain grids. Numerous applications of the codes for the generation of numerical grids with individual and balanced properties in surfaces and domains, in particular, in the tokamak-edge region are demonstrated. The new edition also boasts examples of the implementations of the grid generation codes in the codes for the numerical investigations of gas-dynamics and magnetized plasmas problems.

Grid technology, which has had a significant impact on the efficiency of numerical codes, remains a rapidly advancing field of computational physics and applied mathematics. New achievements are being added by the creation of more sophisticated techniques, modification of the available methods, and implementation of more subtle tools as well as the results of the theories of differential equations, calculus of variations, and Riemannian geometry in the formulation of grid models and analysis of grid properties.

The development of comprehensive differential and variational grid generation techniques reviewed in the monographs of J.F. Thompson, Z.U.A. Warsi, and C.W. Mastin, P. Knupp, and S. Steinberg, and V.D. Liseikin has been largely based on a popular concept in accordance with which a grid model realizing the required grid properties should be formulated through a linear combination of basic and control grid operators with weights. A typical, basic grid operator is the operator responsible for the well-posedness of the grid model and construction of unfolding grids, e.g., the Laplace equations (generalized Laplace equations referred also to as second-order Beltrami equations) or the function of grid smoothness, which produces fixed non-folding grids while grid clustering is controlled by source terms in differential grid formulations or by an adaptation function in variational models. However, such a formulation does not obey the fundamental invariance laws with respect to parameterizations of physical geometries and frequently results in cumbersome governing grid equations. Besides this, the choice of the weight and control functions for providing well-posedness, grid non-degeneracy, and adaptation is largely based on unreliable theoretical assumptions borrowed from one-dimensional models.

The current book revises this popular concept and pursues a more updated and somewhat revolutionary one based on the general fact that an arbitrary one-to-one, smooth multidimensional coordinate transformation deriving a numerical grid in a domain or on a surface is realized by a solution of a system of the Beltrami equations in a suitable monitor metric specified in the physical geometry. The system can be interpreted as the multidimensional equidistribution principle in which the monitor metric tensor is an extension of a scalar-valued weight function. With this interpretation for a mathematical model for generating grids in domains or on surfaces, one need only choose the Beltrami equations, without any complementary control operators that worsen the model, while the required grid properties are realized through the specification of suitable metric tensors.

Thus the single Beltrami mathematical model provides a real foundation for the solution of the challenging problem of the development of comprehensive grid generators. Consequently the efforts of research should be directed towards implementing this model into grid technology by developing approaches for formulating metrics in physical geometries and establishing necessary relations between them and the required grid properties for the purpose of setting up an adequate control of the grid quality by the choice of the suitable metric.

One natural approach for formulating metric tensors and corresponding tensor-valued weight functions is based on the notion of a monitor surface over the physical geometry that undergoes a gridding process. The monitor surface is defined as the graph of some (in general vector-valued) function that takes into account the behavior of the physical solution. This monitor surface, having an inherent metric tensor that can be considered as the very tensor-valued weight function, is suitable for generating adaptive grids with the use of a smoothness functional (which is the functional of energy) whose Euler–Lagrange equations are, in fact, equivalent to the Beltrami equations in the metric of the monitor surface. The resulting grid derived by this metric tends to cluster its nodes in the zones of the large gradient of the function. The approach for formulating the adaptive metric is readily extended to define more general monitor metrics in domains or on surfaces, thus turning them into Riemannian manifolds whose implementation in grid technology allows one to generate grids satisfying the most broad mesh quality requirements.

In order to control the required grid properties by the monitor metrics, one needs a knowledge of geometric characteristics of the monitor geometries and their relations to the resulting grid behavior. This knowledge can be attained with the aid of the theory of multidimensional differential geometry of Riemannian manifolds adjusted to the features of grid technology. The theory of multidimensional differential geometry is really one of the most promising branches of the pure mathematical field of science, capable of pushing grid technology to a more advanced level in its development. Indeed, many notions and characteristics of common surfaces, such as metric tensors, their invariants, first and second fundamental forms, curvatures and torsions of lines, the mean and Gauss curvatures, and Christoffel symbols, have already been used by many authors as natural elements in defining grid quality measures and formulating appropriate variational and differential grid techniques in a unified manner regardless of the geometry of the physical domains and surfaces. A theory of more general geometric objects, such as regular multidimensional surfaces and Riemannian manifolds implemented for generating grids with necessary properties, is expected to become a highly beneficial tool for boosting grid technology. The known relations and techniques of differential geometry also present an efficient means for transforming and modernizing the physical and grid equations into a suitable form. It is presumable that the science of differential geometry will play in numerical grid technology the same role played by the science of matrices in the theory of difference approximations of boundary value problems.

Therefore, there is a need for a monograph that is essentially aimed at providing deep and balanced insight into the fields of grid science, multidimensional geometry adjusted to grid technology, and up-to-date achievements of the applications of geometric tools to the creation of advanced grid techniques. With this background the reader will be able to formulate and develop well-posed grid models and algorithms and analyze grid properties with geometry related tools, thus taking part in the solution of the very challenging problem of the development of advanced comprehensive grid generators.

This monograph gives an account of the geometrization of popular comprehensive grid methods and presents an important extension to the methods related to the application of the technique of Riemannian manifolds to the formulation of grid equations by developing some procedures for the construction of monitor metric tensors. Contrary to classical geometric studies, which center on geometric features and characteristics of specified Riemannian manifolds, the problem of finding appropriate monitor metrics for producing grid systems with the required properties is somewhat an inverse problem of the creation of Riemannian manifolds with desirable geometric characteristics. In accordance with the concept of the inverse problem, the author of the monograph discusses rather thoroughly some new techniques aimed at the construction of special monitor metrics in physical geometries. The techniques are designed by generalizing the projection approach in which the monitor metric in an n-dimensional physical geometry is borrowed from a natural metric of the n-dimensional surface derived by a height monitor function over the geometry. This technology allows the required metric to be defined through the original metric of the physical geometry and certain vector-valued functions.

The book establishes and reviews some of the relations of the Riemannian geometry for the purpose of obtaining new equations with implemented metric characteristics aimed at facilitating the control of the generation of grids with the required properties. Taking advantage of the relations established, the author has converted the equations into a compact form convenient for numerical treatment via the available algorithms.

The technique of multidimensional differential geometry is also applied to study the qualitative effect of a general class of monitor metrics on the resulting mesh. For this purpose a new characteristic of grid clustering is formulated. Certain relations between this measurement and some geometric characteristics of grid hypersurfaces and the monitor functions forming the monitor metrics are established. The well-known results for grids generated by inverted Laplace equations about node-clustering near concave boundary segments of domains and node-rarefaction near convex boundary segments are, using these relations, extended to arbitrary boundary segments and to more general Beltrami equations in monitor metrics. On the basis of the established formulas, the monitor functions are readily estimated in the inverted diffusion and Beltrami grid equations to provide grid clustering or, if it is reasonable, grid rarefaction near arbitrary segments of physical geometries.

Some relations of the mean curvature of the monitor surfaces to the Beltrami equations for grid generation are exhibited. The book also includes a chapter devoted to the implementation of the comprehensive grid equations and the energy functional into numerical codes and to the application of the codes to the numerical solution of some gas-dynamics and plasma-related problems.

Since grid technology has widespread applications to nearly all field problems, this monograph will be useful for a broad range of readers, including teachers, students, and researchers as well as practitioners in applied mathematics, mechanics, biology, medicine, and physics interested in the numerical analysis of multidimensional field problems with complicated geometries and complex solutions.

The book is divided into two parts. Part I of the book gives a geometric background needed for the development of grid generators. The grid equations, codes, and applications are described in Part II.

Part I of the monograph includes Chaps. 1–4. Chapter 1 gives a general introduction to the subject of numerical grids and methods of their generation. Chapters 2–4 introduce the reader to multidimensional differential geometry for the purpose of better understanding those of its techniques that are suitable for the implementation into advanced grid generation technologies. The geometric implementation in grid technology pursued in the book assumes the development of robust techniques for producing appropriate monitor metrics over both physical domains and surfaces thus converting them into Riemannian manifolds. The metrics should guarantee generation of grids with the necessary properties through popular mathematical models.

Part II of the book is devoted to the implementation of geometric tools into the development of grid techniques and codes. It contains Chaps. 5–7. Chapter 5 deals with fundamental elliptic grid models formulated through the operators of Beltrami and diffusion and establishes compact formulas of monitor metrics. Two-dimensional Beltrami equations in the natural metric of a physical surface were originally proposed by Warsi for generating fixed grids on the surface. The ordinary Laplace equations that are the Beltrami equations in the Euclidean metric were applied to generate fixed grids in domains by Crowley and Winslow. One justification of the Beltramian operator is demonstrated in Chap. 5 by the proof of the statement that an arbitrary nondegenerate smooth transformation of a physical domain or surface is realized as a solution of the Dirichlet boundary value problem for the system of Beltrami grid equations in some appropriate metric. The chapter also discusses some variational and harmonic interpretations of the Beltrami equations, in particular, a variational approach for generating harmonic maps through the minimization of energy functionals, which was suggested by Dvinsky.

With the help of the geometric relations, established in Chap. 4, the grid equations introduced in Chap. 5 are transformed in Chap. 6 to equations in invariant forms with respect to independent logical variables. Special monitor metrics over two-dimensional surfaces are designed that result in simpler transformed equations, even in comparison with the equations that have been used for generating fixed grids. The chapter also establishes relations between the monitor functions and geometric characteristics of the Riemannian manifolds produced and the coordinate lines and surfaces generated by a corresponding mathematical model, for the purpose of realization of grid control through a suitable specification of the monitor functions.

Chapter 7 gives a description of some computational codes for generating grids with the numerical solution in the logical domain of the elliptic equations obtained in Chap. 6 by changing mutually dependent and independent variables in the original Beltrami and diffusion equations. Some numerical aspects related to the development of grid generation codes are reviewed, in particular, the application of layer-type functions to formulating monitor metrics and description of two techniques for generating smooth block-structured grids. Numerical results related to the application of the grid technology advocated in the book to some gas-dynamics and plasma problems are also exhibited in this chapter.

The book ends with a list of references.

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