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Albrecht Böttcher · Bernd Silbermann

Analysis of Toeplitz Operators

Second Edition

Prepared jointly with Alexei Karlovich

With 20 Figures

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Preface to the Second Edition

Since the late 1980s, Toeplitz operators and matrices have remained a field of extensive research and the development during the last nearly twenty years is impressive. One encounters Toeplitz matrices in plenty of applications on the one hand, and Toeplitz operators confirmed their role as the basic elementary building blocks of more complicated operators on the other.

Several monographs on Toeplitz and Hankel operators were written during the last decade. These include Peller's grandiose book on Hankel operators and their applications and Nikolski's beautiful easy reading on operators, functions, and systems, with emphasis on topics connected with the names of Hardy, Hankel, and Toeplitz. They also include books by the authors together with Hagen, Roch, Yu. Karlovich, Spitkovsky, Grudsky, and Rabinovich. Thus, results, techniques, and developments in the field of Toeplitz operators are now well presented in the monographic literature. Despite these competitive works, we felt that large parts of the first edition of the present monograph - which is meanwhile out of stock - have not lost their fascination and relevance. Moreover, the first edition has received a warm reception by many colleagues and became a standard reference. This encouraged us to venture on thinking about a second edition, and we are grateful to the Springer Publishing House for showing an interest in this.

The present book is a genuine second edition, which means that it differs from the original version but that it is not a completely new book. We left everything of the first edition, even in the cases where we felt that the material nowadays is not of the same brisance as two decades ago. However, we tried to incorporate as much as possible of the body of knowledge that has grown since about 1990. It is clearly impossible to take into account even only a moderate piece of all the developments in Toeplitz operators during the last ten or fifteen years. We therefore focussed our attention upon topics which are intimately related to those considered in the previous edition or on which we ourselves have worked some time. Many recent achievements were included together with a short comment into the list of references, some others, such as pseudospectra, phenomena caused by SAP symbols, or recent work on

Toeplitz and Wiener-Hopf determinants received own new sections in the text, partly with full proofs.

We hope the present second edition will be welcomed by those who had no chance of purchasing a copy of the original edition and also by all those who are looking for a first or refreshed orientation about the present state of the art in the field.

Braga and Chemnitz,
December 2005

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Preface to the First Edition

This book was originally intended as an extended version of our book "Invertibility and Asymptotics of Toeplitz Matrices", which appeared in 1983. We planned to discuss several topics in more detail, but our main concern was to incorporate a whole series of new results obtained during the last few years. However, we soon realized that the program we had in mind required new thoughts from both the methodological and substantial points of view, and so we decided to attempt writing a completely new book on the analysis of Toeplitz operators.

There are at least two reasons for the continuous and increasing interest in Toeplitz operators. On the one hand, Toeplitz operators are of importance in connection with a variety of problems in physics, probability theory, information and control theory, and several other fields. Although we shall not embark on these problems, the selection of the material of the book is to a certain extent determined by such applications. On the other hand, besides the differential operators, Toeplitz operators constitute one of the most important classes of non-selfadjoint operators and they are a fascinating example of the fruitful interplay between such topics as operator theory, function theory, and the theory of Banach algebras. One main purpose of this book is to elucidate some of the ideas and methods illustrating just the latter aspect.

The theory of Toeplitz operators is a very wide area and even a huge monograph can deal with only some selected topics. Our emphasis is on Toeplitz operators over the circle and over the torus (or, what is the same, discrete Wiener-Hopf operators over the half-axis and over the quarter-plane) viewed as concrete operators on concrete Banach spaces, and a central problem is to establish a relation between the functional-analytic properties of Toeplitz operators and the geometric properties of their symbols. The selection of the special topics has been determined by our own interests and competence. However, having chosen a topic, we try to present it in such a way that it may be taken as a systematic, exhaustive, and modern introduction to the well-known and by now classical results as well as a readable account of some recent developments. A glimpse of the table of contents provides an overall view of the material covered by the book. We merely want to add the following remarks.

Chapter 1 contains a series of notations and definitions. The reader need not study this chapter very carefully; it suffices to glance through it, pick up some notations, and backtrack whenever the necessity arises.

In Chapter 2 we begin by stating elementary properties of Toeplitz operators and finish by proving some of their rather deep-lying properties. This chapter mainly incorporates those results whose proof needs almost no "theory." Moreover, it may be viewed as the trunk of a tree, the boughs and twigs of which are the concern of the forthcoming chapters.

In Chapter 3 we start preparing the more delicate theory of Toeplitz operators. It is devoted to matrix functions which are locally sectorial in a very sensitive sense. This chapter also includes the study of the phenomenon of the asymptotic multiplicativity of approximate identities and Sarason's theory of piecewise quasicontinuous functions. In this chapter we also devise and give a first application of some sort of machinery (or "philosophy") that will be employed repeatedly in the remaining chapters: algebraization, essentialization, localization, determination of local spectra.

Chapter 4 is concerned with the Hilbert space theory of block Toeplitz operators. There we prove Fredholmness and compute the index of Toeplitz operators whose symbol is locally sectorial over QC, describe Axler's transfinite localization approach to maximal antisymmetric sets for $C+H^\infty$, present the theory of local Toeplitz operators due to Douglas, Clancey, Gosselin, study symbols with a specified local range (in particular, symbols with two or three essential cluster points), and develop a new approach to algebras generated by Toeplitz operators and related objects.

Chapters 5 and 6 deal with block Toeplitz operators on weighted H^p and ℓ^p spaces, respectively. Ours is, to a great extent, a novel presentation of these topics. We provide new proofs of the classical results of the well-known monographs by Gohberg, Feldman and Gohberg, Krupnik, and we incorporate numerous results which are only known from mathematical journals, primarily Soviet ones.

Chapter 7 is a self-contained and up to date theory of finite section method (reduction method) for Toeplitz operators. We prove that the finite section method is applicable to block Toeplitz operators on H^2 with symbols that are locally sectorial over QC, we develop a sufficiently simple theory for operators with piecewise continuous symbols on H^p and ℓ^p, we study symbols with singularities of Fisher-Hartwig type, and we conclude by proving the very recent and noteworthy result of Treil, according to which there are invertible Toeplitz operators on H^2 to which the finite section method is not applicable.

Chapter 8 is a comprehensive treatment of Toeplitz operators over the quarter-plane and is, at least to a certain degree, a novelty in the monographic literature. We emphasize that we study both the Fredholm theory and the theory of finite section methods of quarter-plane operators with discontinuous symbols.

Chapter 9 looks at Wiener-Hopf integral operators. There we point out the common features between Wiener-Hopf integral operators and their discrete analogues (the Toeplitz operators) but also dwell on the significant differences between these two classes of operators. In this chapter we also consider operators with almost periodic, semi-almost periodic, and other kinds of oscillating symbols.

Chapter 10 is a systematic and self-contained theory of Toeplitz determinants. The material presented ranges from the classical Szegő-Widom limit theorems to a proof of the conjecture of Fisher-Hartwig for some important special cases. We shall demonstrate that the very attractive field of Toeplitz determinants requires results from all foregoing chapters and may thus serve as a beautiful application of the functional analysis of Toeplitz operators. In particular, we shall show that some important problems on Toeplitz determinants can be solved by working with Toeplitz operators on the spaces $H^2(\varrho)$ and ℓ^p_μ, so that passage from the Hilbert space theory to the Banach space theory does not turn out to be a purely academic matter.

Let us also point out three peculiarities of the present monograph.

First, Banach algebra techniques combined with local principles are our main tool for tackling Toeplitz operators. That such methods can be successfully applied to the study of the Fredholm theory of Toeplitz operators is well-known from Douglas' book. However, this approach has only recently proved to be a powerful technique of studying projection methods, harmonic approximation, or stable convergence (and thus index computation) for Toeplitz operators. Moreover, our consistent use of local Banach algebra technique will not only provide a unified technique of solving various problems related to Toeplitz operators, but will allow us to reformulate many classical results in pretty nice language. For instance, the well-known result that a Toeplitz operator with piecewise continuous symbol is Fredholm on H^p or ℓ^p if and only if the curve obtained from the essential range of its symbol by filling in certain circular arcs does not contain the origin reads in this language as follows: the local spectrum of the operator is either a point or a certain circular arc.

Secondly, we shall consider Toeplitz operators on the (Hilbert) space $H^2 \cong \ell^2$ and on the weighted (Banach) spaces $H^p(\varrho)$ and ℓ^p_μ. Each of these three situations has its peculiarities and requires its own techniques. While there are excellent and comprehensive discussions of the Hilbert space theory in the well-known monographs by Gohberg and Feldman, Douglas, and Nikolski, the same cannot be said about Banach space theory. We hope that the present book will fill this gap and, moreover, will also make a series of new contributions to the Hilbert space case.

Thirdly, it should be explicitly noticed that our emphasis is on matrix-valued symbols. Many problems on Toeplitz operators have fairly fast solution in the scalar case, whereas substantial difficulties arise in the matrix case.

Finally, to see what this book is all about, it should also be mentioned that the following topics are not touched upon: Toeplitz operators on domains, balls, or manifolds; pseudodifferential operators; Breuer-Fredholm results and generalized index theory; operator-valued symbols; invariant subspaces; linear algebra and computational mathematics of finite Toeplitz matrices. This list is naturally incomplete. Let it also be understood that this is a book on Toeplitz operators and not on Hankel or singular integral operators, although we pay due attention to these two classes of operators.

We ventured on writing a book for both the beginner and the specialist. In the beginner's interest we gave a rather full description of those topics which form the background to the theory of Toeplitz operators (e.g., some students of ours acknowledged the rather lengthy explanation of what a "fiber" is). We also provided the bulk of results with detailed proofs. We did this in order to teach the reader not only the results but also (or mainly) the techniques for proving them. This is, of course, not always in the beginner's interest, but we hope the specialist will relish some details of these proofs. Some of the results and techniques are new and published here for the first time and are thus primarily addressed to the specialist. Many results are taken from the periodicals and are first cited with detailed proofs.

We included a series of problems which we declared to be "open". Some of them are well-known as open problems, some others are merely open in the sense that *we* have not found a solution within a few hours, days or weeks. In either case we followed the policy that we would better confess our own inability than hide something.

We made the attempt of supplying all results with a source; however, the evolution of many theorems involves too many contributors, and so it may occur that our reference is not the right one. We hope that the reader will excuse our faulty referencing and we accept any criticism in this direction. Finally, we have labelled a lemma or theorem only when a name seems to have been attached to it by common usage.

We wish to express our sincere appreciation to our colleagues Roland Hagen and Steffen Roch, who both read the bulk of the manuscript very carefully and eradicated not only a large number of mistakes and (sometimes serious) errors but helped with their criticism to essentially improve the book. We would also like to thank Mrs. Marianne Graupner and Mrs. Isolde Scholz for all the trouble they took in typing the entire manuscript. Finally, we are pleased to express our gratitude to the Akademie-Verlag Publishing House, especially to the Editor, Dr. Reinhard Höppner, for inviting us to write this monograph and for the careful performance of the book.

Special Acknowledgement. In May and June 1987, Naum Krupnik visited the Chemnitz University of Technology. During these two months he read the whole manuscript with great enthusiasm and made a large number of valuable remarks, a major part of which could still be incorporated into the text. We are extremely grateful to him for improving the book by his uncommon expertise.

Chemnitz, 1987 *Albrecht Böttcher, Bernd Silbermann*

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