Elements for Physics

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Quantities, Qualities, and Intrinsic Theories

With 44 Figures (10 in colour)

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To Maria

Preface

Physics is very successful in describing the world: its predictions are often impressively accurate. But to achieve this, physics limits terribly its scope. Excluding from its domain of study large parts of biology, psychology, economics or geology, physics has concentrated on quantities, i.e., on notions amenable to accurate measurement.

The meaning of the term physical 'quantity' is generally well understood (everyone understands what it is meant by "the frequency of a periodic phenomenon", or "the resistance of an electric wire"). It is clear that behind a set of *quantities* like temperature − inverse temperature − logarithmic temperature, there is a qualitative notion: the 'cold−hot' *quality*. Over this one-dimensional quality space, we may choose different 'coordinates': the temperature, the inverse temperature, etc. Other quality spaces are multidimensional. For instance, to represent the properties of an ideal elastic medium we need 21 coefficients, that can be the 21 components of the elastic stiffness tensor $c_{ijk\ell}$, or the 21 components of the elastic compliance tensor (inverse of the stiffness tensor), or the proper elements (six eigenvalues and 15 angles) of any of the two tensors, etc. Again, we are selecting coordinates over a 21-dimensional quality space. On this space, each point represents a particular elastic medium.

So far, the consideration is trivial. What is important is that *it is always possible to define the distance between two points of any quality space, and this distance is —inside a given theoretical context— uniquely defined*. For instance, two periodic phenomena can be characterized by their periods, T_1 and T_2 , or by their frequencies, ν_1 and ν_2 . The only definition of distance that respects some clearly defined invariances is $D = |\log(T_2/T_1)| = |\log(\nu_2/\nu_1)|$.

For many vector and tensor spaces, the distance is that associated with the ordinary norm (of a vector or a tensor), but some important spaces have a more complex structure. For instance, 'positive tensors' (like the electric permittivity or the elastic stiffness) are not, in fact, elements of a linear space, but oriented geodesic segments of a curved space. The notion of *geotensor* ("geodesic tensor") is developed in chapter 1 to handle these objects, that are like tensors but that do not belong to a linear space.

The first implications of these notions are of mathematical nature, and a point of view is proposed for understanding Lie groups as metric manifolds with curvature and torsion. On these manifolds, a sum of geodesic segments can be introduced that has the very properties of the group. For instance, in the manifold representing the group of rotations, a 'rotation vector' is not a vector, but a geodesic segment of the manifold, and the composition of rotations is nothing but the geometric sum of these segments.

More fundamental are the implications in physics. As soon as we accept that behind the usual physical quantities there are quality spaces, that usual quantities are only special 'coordinates' over these quality spaces, and that there is a metric in each space, the following question arises: Can we do physics intrinsically, i.e., can we develop physics using directly the notion of physical quality, and of metric, and without using particular coordinates (i.e., without any particular choice of physical quantities)? For instance, Hooke's law $\sigma_{ij} = c_{ij}{}^{k\ell}\,\varepsilon_{k\ell}$ is written using three quantities, stress, stiffness, and strain. But why not using the exponential of the strain, or the inverse of the stiffness? One of the major theses of this book is that physics can, and must, be developed independently of any particular choice of coordinates over the quality spaces, i.e., independently of any particular choice of physical quantities to represent the measurable physical qualities.

Most current physical theories, can be translated so that they are expressed using an intrinsic language. Other theories (like the theory of linear elasticity, or Fourier's theory of heat conduction) cannot be written intrinsically. I claim that these theories are inconsistent, and I propose their reformulation.

Mathematical physics strongly relies on the notion of derivative (or, more generally, on the notion of tangent linear mapping). When taking into account the geometry of the quality spaces, another notion appears, that of *declinative*. Theories involving nonflat manifolds (like the theories involving Lie group manifolds) are to be expressed in terms of declinatives, not derivatives. This notion is explored in chapter 2.

Chapter 3 is devoted to the analysis of some spaces of physical qualities, and attempts a classification of the more common types of physical quantities used on these spaces. Finally, chapter 4 gives the definition of an intrinsic physical theory and shows, with two examples, how these intrinsic theories are built.

Many of the ideas presented in this book crystallized during discussions with my colleagues and students. My friend Bartolomé Coll deserves special mention. His understanding of mathematical structures is very deep. His logical rigor and his friendship have made our many discussions both a pleasure and a source of inspiration. Some of the terms used in this book have been invented during our discussions over a cup of coffee at Café Beaubourg, in Paris. Special thanks go to my professor Georges Jobert, who introduced me to the field of inverse problems, with dedication and rigor. He has contributed to this text with some intricate demonstrations. Another friend, Klaus Mosegaard, has been of great help, since the time we developed

together Monte Carlo methods for the resolution of inverse problems. With probability one, he defeats me in chess playing and mathematical problem solving. Discussions with Peter Basser, João Cardoso, Guillaume Evrard, Jean Garrigues, José-Maria Pozo, John Scales, Loring Tu, Bernard Valette, Peiliang Xu, and Enrique Zamora have helped shape some of the notions presented in this book.

Paris, August 2005 *Albert Tarantola*

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Overview

One-dimensional Quality Spaces

Consider a one-dimensional space, each point $\mathcal{N}$ of it representing a musical note. This line has to be imagined infinite in its two senses, with the infinitely acute tones at one "end" and the infinitely grave tones at the other "end". Musicians can immediately give the *distance* between two points of the space, i.e., between two notes, using the octave as unit. To express this distance by a formula, we may choose to represent a note by its frequency, ν, or by its period, τ. The distance between two notes $\mathcal{N}_1$ and $\mathcal{N}_2$ is[1]

$$D_{\mathrm{music}}(\mathcal{N}_1, \mathcal{N}_2) \;=\; |\,\log_2 \frac{\nu_2}{\nu_1}\,| \;=\; |\,\log_2 \frac{\tau_2}{\tau_1}\,| \quad . \tag{1}$$

This distance is the only one that has the following properties:

- its expression is identical when using the positive quantity $\nu = 1/\tau$ or its inverse, the positive quantity $\tau = 1/\nu$;
- it is additive, i.e., for any set of three ordered points $\{\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3\}$, the distance from point $\mathcal{N}_1$ to point $\mathcal{N}_2$, plus the distance from point $\mathcal{N}_2$ to point $\mathcal{N}_3$, equals the distance from point $\mathcal{N}_1$ to point $\mathcal{N}_3$.

This one-dimensional space (or, to be more precise, this one-dimensional *manifold*) is a simple example of a *quality space*. It is a metric manifold (the distance between points is defined). The quantities frequency ν and period τ are two of the *coordinates* that can be used on the quality space of the musical notes to characterize its points. Infinitely many more coordinates are, of course, possible, like the logarithmic frequency $\nu^* = \log(\nu/\nu_0)$, the cube of the frequency, $\eta = \nu^3$, etc. Given the expression for the distance in some coordinate system, it is easy to obtain an expression for it using another coordinate system. For instance, it follows from equation (1) that the distance between two musical notes is, in terms of the logarithmic frequency, $D_{\mathrm{music}}(\mathcal{N}_1, \mathcal{N}_2) = |\,\nu^*_2 - \nu^*_1\,|$.

There are many quantities in physics that share three properties: (*i*) their range of variation is $(0, \infty)$, (*ii*) they are as commonly used as their in-

[1] To obtain the distance in octaves, one must use base 2 logarithms.

verses, and (*iii*) they display the Benford effect.[2] Examples are the frequency ($\nu = 1/\tau$) and period ($\tau = 1/\nu$) pair, the temperature ($T = 1/\beta$) and thermodynamic parameter ($\beta = 1/T$) pair, or the resistance ($R = 1/C$) and conductance ($C = 1/R$) pair. These quantities typically accept the expression in formula (1) as a natural definition of distance. In this book we say that we have a pair of *Jeffreys quantities*.

For instance, before the notion of *temperature*[3] was introduced, physicists followed Aristotle in introducing the *cold−hot (quality) space*. Even if a particular coordinate over this one-dimensional manifold was not available, physicists could quite precisely identify many of its points: the point $\mathcal{Q}_1$ corresponding to the melting of sulphur, the point $\mathcal{Q}_2$ corresponding to the boiling of water, etc. Among the many coordinates today available in the cold−hot space (like the Celsius or the Fahrenheit temperatures), the pair absolute temperature $T = 1/\beta$ and thermodynamic parameter $\beta = 1/T$ are obviously a Jeffreys pair. In terms of these coordinates, the natural distance between two points of the cold−hot space is (using natural logarithms)

$$D_{\text{cold−hot}}(\mathcal{Q}_1, \mathcal{Q}_2) \;=\; |\, \log \frac{T_2}{T_1} \,| \;=\; |\, \log \frac{\beta_2}{\beta_1} \,| \;=\; |\, T^*_2 - T^*_1 \,| \;=\; |\, \beta^*_2 - \beta^*_1 \,| \quad , \tag{2}$$

where, for more completeness, the logarithmic temperature T^* and the logarithmic thermodynamic parameter β^* have also been introduced. An expression using other coordinates is deduced using any of those equivalent expressions. For instance, using Celsius temperatures, $D_{\text{cold−hot}}(\mathcal{Q}_1, \mathcal{Q}_2) = |\, \log(\, (t_2 + T_0)/(t_1 + T_0)\,)\, |$, where $T_0 = 273.15\,\text{K}$.

At this point, without any further advance in the theory, we could already ask a simple question: if the tone produced by a musical instrument depends on the position of the instrument in the cold−hot space (using ordinary language we would say that the 'frequency' of the note depends on the 'temperature', but we should not try to be specific), what is the simplest dependence that we can imagine? Surely a *linear* dependence. But as both spaces, the space of musical notes and the cold−hot space, are metric, the only *intrinsic* definition of linearity is a proportionality between the *distances* in the two spaces,

$$D_{\text{music}}(\mathcal{N}_1, \mathcal{N}_2) \;=\; \alpha \, D_{\text{cold−hot}}(\mathcal{Q}_1, \mathcal{Q}_2) \quad , \tag{3}$$

where α is a positive real number. Note that we have just expressed a physical law without being specific about the many possible physical quantities

[2]The Benford effect is an uneven probability distribution for the first digit in the numerical expression of a quantity: when using a base K number system, the probability that the first digit is n is $p_n = \log_K (n+1)/n$. For instance, in the usual base 10 system, about 30% of the time the first digit is one, while for only 5% of the time is the first digit a nine. See details in chapter 3.

[3]Before Galileo, the quantity 'temperature' was not defined. Around 1592, he invented the first thermometer, using air.

that one may use in each of the two quality spaces. Choosing, for instance, temperature T in the cold−hot space, and frequency ν in the space of musical notes, the expression for the linear law (3) is

$$\nu_2 / \nu_1 = (T_2 / T_1)^\alpha \quad . \tag{4}$$

Note that the linear law takes a *formally* linear aspect only if logarithmic frequency (or logarithmic period) and logarithmic temperature (or logarithmic thermodynamic parameter) are used. An expression like $\nu_2 - \nu_1 = \alpha\,(T_2 - T_1)$ although formally linear, is not a linear law (as far as we have agreed on given metrics in our quality spaces).

Multi-dimensional Quality Spaces

Consider a homogeneous piece of a linear elastic material, in its unstressed state. When a (homogeneous) stress $\boldsymbol{\sigma} = \{\sigma^{ij}\}$ is applied, the body experiences a strain $\boldsymbol{\varepsilon} = \{\varepsilon^{ij}\}$ that is related to the stress through any of the two equivalent equations (Hooke's law)

$$\varepsilon^{ij} = d^{ij}{}_{k\ell}\,\sigma^{k\ell} \qquad ; \qquad \sigma^{ij} = c^{ij}{}_{k\ell}\,\varepsilon^{k\ell} \quad , \tag{5}$$

where $\mathbf{d} = \{d^{ij}{}_{k\ell}\}$ is the *compliance* tensor, and $\mathbf{c} = \{c^{ij}{}_{k\ell}\}$ is the *stiffness* tensor. These two tensors are positive definite and are mutually inverse, $d^{ij}{}_{k\ell}\,c^{k\ell}{}_{mn} = c^{ij}{}_{k\ell}\,d^{k\ell}{}_{mn} = \delta^i_m\,\delta^j_n$, and one can use any of the two to characterize the elastic medium.

In elementary elasticity theory one assumes that the compliance tensor has the symmetries $d_{ijk\ell} = d_{jik\ell} = d_{k\ell ij}$, with an equivalent set of symmetries for the stiffness tensor. An easy computation shows that (in 3D media) one is left with 21 degrees of freedom, i.e., 21 quantities are necessary and sufficient to characterize a linear elastic medium. We can then introduce an abstract 21-dimensional manifold $\mathfrak{E}$, such that each point $\mathcal{E}$ of $\mathfrak{E}$ corresponds to an elastic medium (and vice versa). This is the (quality) space of elastic media.

Which sets of 21 quantities can we choose to represent a linear elastic medium? For instance, we can choose 21 independent components of the compliance tensor $d^{ij}{}_{k\ell}$, or 21 independent components of the stiffness tensor $c^{ij}{}_{k\ell}$, or the six eigenvalues and the 15 proper angles of the one or the other. Each of the possible choices corresponds to choosing a coordinate system over $\mathfrak{E}$.

Is the manifold $\mathfrak{E}$ metric, i.e., is there a natural definition of distance between two of its points? The requirement that the distance must have the same expression in terms of compliance, $\mathbf{d}$, and in terms of stiffness, $\mathbf{c}$, that it must have an invariance of scale (multiplying all the compliances or all the stiffnesses by a given factor should not alter the distance), and that it should depend only on the invariant scalars of the compliance or of the stiffness tensor leads to a unique expression. The distance between the elastic medium $\mathcal{E}_1$, characterized by the compliance tensor $\mathbf{d}_1$ or the stiffness tensor $\mathbf{c}_1$, and the elastic medium $\mathcal{E}_2$ characterized by the compliance tensor $\mathbf{d}_2$ or the stiffness tensor $\mathbf{c}_2$, is

$$D_{\mathfrak{E}}(\mathcal{E}_1, \mathcal{E}_2) \;=\; \|\; \log(\mathbf{d}_2\, \mathbf{d}_1^{-1}) \;\| \;=\; \|\; \log(\mathbf{c}_2\, \mathbf{c}_1^{-1}) \;\| \quad . \tag{6}$$

In this equation, the logarithm of an adimensional, positive definite tensor $\mathbf{T} = \{T^{ij}{}_{k\ell}\}$ can be defined through the series[4]

$$\log \mathbf{T} \;=\; (\mathbf{T} - \mathbf{I}) - \tfrac{1}{2}\,(\mathbf{T} - \mathbf{I})^2 + \ldots \quad . \tag{7}$$

Alternatively, the logarithm of an adimensional, positive definite tensor can be defined as the tensor having the same proper angles as the original tensor, and whose eigenvalues are the logarithms of the eigenvalues of the original tensor. Also in equation (6), the norm of a tensor $\mathbf{t} = \{t^{ij}{}_{k\ell}\}$ is defined through

$$\|\, \mathbf{t} \,\| \;=\; \sqrt{t^{ij}{}_{k\ell}\, t^{k\ell}{}_{ij}} \quad . \tag{8}$$

It can be shown (see chapter 1) that the finite distance in equation (6) does derive from a metric, in the sense of the term in differential geometry, i.e., it can be deduced from a quadratic expression defining the distance element ds^2 between two infinitesimally close points.[5] An immediate question arises: is this 21-dimensional manifold flat? To answer this question one must evaluate the Riemann tensor of the manifold, and when this is done, one finds that this tensor is different from zero: *the manifold of elastic media has curvature*.

Is this curvature an artefact, irrelevant to the physics of elastic media, or is this curvature the sign that the quality spaces here introduced have a non-trivial geometry that may allow a geometrical formulation of the equations of physics? This book is here to show that it is the second option that is true. But let us take a simple example: the three-dimensional rotations.

A rotation $\mathcal{R}$ can be represented using an orthogonal matrix $\mathbf{R}$. The *composition* of two rotations is defined as the rotation $\mathcal{R}$ obtained by first applying the rotation $\mathcal{R}_1$, then the rotation $\mathcal{R}_2$, and one may use the notation

$$\mathcal{R} \;=\; \mathcal{R}_2 \circ \mathcal{R}_1 \quad . \tag{9}$$

It is well known that when rotations are represented by orthogonal matrices, the composition of two rotations is obtained as a matrix product:

$$\mathbf{R} \;=\; \mathbf{R}_2\, \mathbf{R}_1 \quad . \tag{10}$$

But there is a second useful representation of a rotation, in terms of a rotation pseudovector $\boldsymbol{\rho}$, whose axis is the rotation axis and whose norm equals the

[4]I.e., $(\log \mathbf{T})^{ij}{}_{k\ell} = (T^{ij}{}_{k\ell} - \delta^i_k\, \delta^j_\ell) - \frac{1}{2}\,(T^{ij}{}_{rs} - \delta^i_r\, \delta^j_s)\,(T^{rs}{}_{k\ell} - \delta^r_k\, \delta^s_\ell) + \ldots$.

[5]This distance is closely related to the "Cartan metric" of Lie group manifolds.

rotation angle. As pseudovectors are, in fact, antisymmetric tensors, let us denote by $\mathbf{r}$ the antisymmetric matrix related to the components of the pseudovector $\boldsymbol{\rho}$ through the usual duality,[6] $r_{ij} = \epsilon_{ijk}\,\rho^k$. For instance, in a Euclidean space, using Cartesian coordinates,

$$\mathbf{r} = \begin{pmatrix} r_{xx} & r_{xy} & r_{xz} \\ r_{yx} & r_{yy} & r_{yz} \\ r_{zx} & r_{zy} & r_{zz} \end{pmatrix} = \begin{pmatrix} 0 & \rho^z & -\rho^y \\ -\rho^z & 0 & \rho^x \\ \rho^y & -\rho^x & 0 \end{pmatrix} \quad . \tag{11}$$

We shall sometimes call the antisymmetric matrix $\mathbf{r}$ the rotation "vector".

Given an orthogonal matrix $\mathbf{R}$ how do we obtain the antisymmetric matrix $\mathbf{r}$? It can be seen that the two matrices are related via the log−exp duality:

$$\mathbf{r} = \log \mathbf{R} \quad . \tag{12}$$

This is a very simple way for obtaining the rotation vector $\mathbf{r}$ associated to an orthogonal matrix $\mathbf{R}$. Reciprocally, to obtain the orthogonal matrix $\mathbf{R}$ associated to the rotation vector $\mathbf{r}$, we can use

$$\mathbf{R} = \exp \mathbf{r} \quad . \tag{13}$$

With this in mind, it is easy to write the composition of rotations in terms of the rotation vectors. One obtains

$$\mathbf{r} = \mathbf{r}_2 \oplus \mathbf{r}_1 \quad , \tag{14}$$

where the operation $\oplus$ is defined, for any two tensors $\mathbf{t}_1$ and $\mathbf{t}_2$, as

$$\mathbf{t}_2 \oplus \mathbf{t}_1 \equiv \log(\exp \mathbf{t}_2 \, \exp \mathbf{t}_1) \quad . \tag{15}$$

The two expressions (10) and (14) are two different representations of the abstract notion of composition of rotations (equation 9), respectively in terms of orthogonal matrices and in terms of antisymmetric matrices (rotation vectors). Let us now see how the operation $\oplus$ in equation (14) can be interpreted as a sum, provided that one takes into account the geometric properties of the space of rotations.

It is well known that the rotations form a group, the Lie group SO(3) . Lie groups are manifolds, in fact, quite nontrivial manifolds, having curvature and torsion.[7] In the (three-dimensional) Lie group manifold SO(3) , the orthogonal matrices $\mathbf{R}$ can be seen as the points of the manifold. When the identity matrix $\mathbf{I}$ is taken as the origin of the manifold, an antisymmetric matrix $\mathbf{r}$ can be interpreted as the oriented geodesic segment going from the origin $\mathbf{I}$ to the point $\mathbf{R} = \exp \mathbf{r}$. Then, let two rotations be represented by the two antisymmetric matrices $\mathbf{r}_2$ and $\mathbf{r}_1$, i.e., by two oriented geodesic

[6] Here, ϵ_{ijk} is the totally antisymmetric symbol.

[7] And such that autoparallel lines and geodesic lines coincide.

segments of the Lie group manifold. It is demonstrated in chapter 1 that the *geometric sum* of the two segments (performed using the curvature and torsion of the manifold) exactly corresponds to the operation $\mathbf{r}_2 \oplus \mathbf{r}_2$ introduced in equations (14) and (15), i.e., *the geometric sum of two oriented geodesic segments of the Lie group manifold* is *the group operation*.

This example shows that the nontrivial geometry we shall discover in our quality spaces is fundamentally related to the basic operations to be performed. One of the major examples of physical theories in this book is, in chapter 4, the theory of ideal elastic media. When acknowledging that the usual 'configuration space' of the body is, in fact, (a submanifold of) the Lie group manifold $GL^+(3)$ (whose 'points' are all the 3×3 real matrices with positive determinant), one realizes that the *strain* is to be defined as a geodesic line joining two configurations: the strain is not an element of a linear space, but a geodesic of a Lie group manifold. This, in particular, implies that the proper definition of strain is *logarithmic*.

This is one of the major lessons to be learned from this book: the tensor equations of properly developed physical theories, usually contain logarithms and exponentials of tensors. The conspicuous absence of logarithms and exponentials in present-day physics texts suggests that there is some basic aspect of mathematical physics that is not well understood. I claim that a fundamental invariance principle should be stated that is not yet recognized.

Invariance Principle

Today, a physical theory is seen as relating different physical quantities. But we have seen that *physical quantities* are nothing but *coordinates* over spaces of *physical qualities*. While present tensor theories assure invariance of the equations with respect to a change of coordinates over the physical space (or the physical space-time, in relativity), we may ask if there is a formulation of the tensor theories that assure invariance with respect to *any* choice of coordinates over *any* space, including the spaces of physical qualities (i.e., invariance with respect to any choice of physical quantities that may represent the physical qualities).

The goal of this book is to demonstrate that the answer to that question is positive.

For instance, when formulating Fourier's law of heat conduction, we have to take care to arrive at an equation that is independent of the fact that, over the cold−hot space, we may wish to use as coordinate the temperature, its inverse, or its cube. When doing so, one arrives at an expression (see equation 4.21) that has no immediate resemblance to the original Fourier's law. This expression does not involve specific quantities; rather, it is valid for any possible choice of them. When being specific and choosing, for instance, the (absolute) temperature T the law becomes

$$\phi_i \;=\; -\kappa\,\frac{1}{T}\,\frac{\partial T}{\partial x^i} \quad , \tag{16}$$

where $\{x^i\}$ is any coordinate system in the physical space, ϕ_i is the heat flux, and κ is a constant. This is not Fourier's law, as there is an extra factor $1/T$. Should we write the law using, instead of the temperature, the thermodynamic parameter $\beta = 1/T$, we would arrive at

$$\phi_i \;=\; +\kappa\,\frac{1}{\beta}\,\frac{\partial \beta}{\partial x^i} \quad . \tag{17}$$

It is the symmetry between these two expressions of the law (a symmetry that is not satisfied by the original Fourier's law) that suggests that the equations at which we arrive when using our (strong) invariance principle may be more physically meaningful than ordinary equations. In fact, nothing in the arguments of Fourier's work (1822) would support the original equation, $\phi_i = -\kappa\,\partial T/\partial x^i$, better than our equation (16). In chapter 4, it is suggested that, quantitatively, equations (16) and (17) are at least as good as Fourier's law, and, qualitatively, they are better.

In the case of one-dimensional quality spaces, the necessary invariance of the expressions is achieved by taking seriously the notion of *one-dimensional linear space*. For instance, as the cold−hot quality space is a one-dimensional metric manifold (in the sense already discussed), once an arbitrary origin is chosen, it becomes a linear space. Depending on the particular coordinate chosen over the manifold (temperature, cube of the temperature), the natural basis (a single vector) is different, and vectors on the space have different components. Nothing is new here with respect to the theory of linear spaces, but this is not the way present-day physicists are trained to look at one-dimensional qualities.

In the case of multi-dimensional quality spaces, one easily understands that physical theories do not relate particular quantities but, rather, they relate the geometric properties of the different quality spaces involved. For instance, the law defining an ideal elastic medium can be stated as follows: when a body is subjected to a linear change of stress, its configuration follows a geodesic line in the configuration space.[8]

Mathematics

To put these ideas on a clear basis, we need to develop some new mathematics.

[8]More precisely, as we shall see, an ideal elastic medium is defined by a 'geodesic mapping' between the (linear) stress space and the submanifold of the Lie group manifold $GL^+(3)$ that is geodesically connected to the origin of the group (this submanifold is the configuration space).

Our quality spaces are manifolds that, in general, have curvature and torsion (like the Lie group manifolds). We shall select an origin on the manifold, and consider the collection of all 'autoparallel' or 'geodesic' segments with that common origin. Such an oriented segment shall be called an *autovector*. The sum of two autovectors is defined using the parallel transport on the manifold. Should the manifold be flat, we would obtain the classic structure of linear space. But what is the structure defined by the 'geometric sum' of the autovectors? When analyzing this, we will discover the notion of *autovector space*, which will be introduced axiomatically. In doing so, we will find, as an intermediary, the *troupe* structure (in short, a group without the associativity property).

With this at hand, we will review the basic geometric properties of Lie group manifolds, with special interest in curvature, torsion and parallel transport. While de-emphasizing the usual notion of Lie algebra, we shall study the interpretation of the group operation in terms of the geometric sum of oriented autoparallel (and geodesic) segments. A special term is used for these oriented autoparallel segments, that of *geotensor* (for "geodesic tensor").

Geotensors play an important role in the theory. For many of the objects called "tensors" in physics are improperly named. For instance, as mentioned above, the strain $\boldsymbol{\varepsilon}$ that a deforming body may experience *is* a geodesic of the Lie group manifold $GL^+(3)$. As such, it is not an element of a linear space, but an element of a space that, in general, is not flat. Unfortunately, this seems to be more than a simple misnaming: the conspicuous absence of the logarithm and the exponential functions in tensor theories suggests that the geometric structure actually behind some of the "tensors" in physics is not clearly understood. This is why a special effort is developed in this text to define explicitly the main properties of the log−exp duality for tensors.

There is another important mathematical notion that we need to revisit: that of derivative. There are two implications to this. First, when taking seriously the tensor character of the derivative, one does not define the derivative of one *quantity* with respect to another *quantity*, but the derivative of one *quality* with respect to another *quality*. In fact, we have already seen one example of this: in equations (18) and (19) the same derivative is expressed using different coordinates in the cold−hot space (the temperature T and the inverse temperature β). This is the very reason why the law of heat conduction proposed in this text differs from the original Fourier's law.

A less obvious deviation from the usual notion of derivative is when the *declinative* of a mapping is introduced. The declinative differs from the derivative in that the geometrical objects considered are 'transported to the origin'. Consider, for instance, a solid rotating around a point. When characterizing the 'attitude' of the body at some instant t by the (orthogonal) rotation matrix $\mathbf{R}(t)$, we are, in fact defining a mapping from the time axis into the rotation group $SO(3)$. The declinative of this mapping happens to be[9]

$$\dot{\mathbf{R}}(t)\,\mathbf{R}(t)^{-1} \quad , \tag{20}$$

where $\dot{\mathbf{R}}$ is the derivative. The expression $\dot{\mathbf{R}}(t)\,\mathbf{R}(t)^{-1}$ gives, in fact, the instantaneous *rotation velocity*, $\boldsymbol{\omega}(t)$. While the derivative produces $\dot{\mathbf{R}}(t)$, that has no simple meaning, the declinative directly produces the rotation velocity $\boldsymbol{\omega}(t) = \dot{\mathbf{R}}(t)\,\mathbf{R}(t)^{-1} = \dot{\mathbf{R}}(t)\,\mathbf{R}(t)^{*}$ (because the geometry of the rotation group $SO(3)$ is properly taken into account).

Contents

While the mathematics concerning the autovector spaces are developed in chapter 1, those concerning derivatives and declinatives are developed in chapter 2. Chapter 3 gives some examples of identification of the quality spaces behind some of the common physical quantities, and chapter 4 develops two special examples of intrinsic physical theories, the theory of heat conduction and the theory of ideal elastic media. Both theories are chosen because they quantitatively disagree with the versions found in present-day texts.

[9] $\dot{\mathbf{R}}(t)\,\mathbf{R}(t)^{-1}$ is different from $(\log \mathbf{R})^{\cdot} \equiv d(\log \mathbf{R})/dt$.