Grundlehren der mathematischen Wissenschaften 334

A Series of Comprehensive Studies in Mathematics

Edoardo Sernesi

Deformations of Algebraic Schemes

Springer

Edoardo Sernesi
Universitá "Roma Tre"
Department of Mathematics
Largo San Leonardo Murialdo 1
00146 Roma, Italy
e-mail: *sernesi@mat.uniroma3.it*

Library of Congress Control Number: 2006924565

Mathematics Subject Classification (2000): 14D15, 14B12

ISSN 0072-7830
ISBN-10 3-540-30608-0 Springer Berlin Heidelberg New York
ISBN-13 978-3-540-30608-5 Springer Berlin Heidelberg New York

Springer is a part of Springer Science+Business Media
springer.com

Printed in The Netherlands

Typesetting: by the author and SPI Publisher Services using a Springer LaTeX macro package
Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper SPIN: 11371335 41/SPI 5 4 3 2 1 0

Preface

In one sense, deformation theory is as old as algebraic geometry itself: this is because all algebro-geometric objects can be "deformed" by suitably varying the coefficients of their defining equations, and this has of course always been known by the classical geometers. Nevertheless, a correct understanding of what "deforming" means leads into the technically most difficult parts of our discipline. It is fair to say that such technical obstacles have had a vast impact on the crisis of the classical language and on the development of the modern one, based on the theory of schemes and on cohomological methods.

The modern point of view originates from the seminal work of Kodaira and Spencer on small deformations of complex analytic manifolds and from its formalization and translation into the language of schemes given by Grothendieck. I will not recount the history of the subject here since good surveys already exist (e.g. [27], [138], [145], [168]). Today, while this area is rapidly developing, a self-contained text covering the basic results of what we can call "classical deformation theory" seems to be missing. Moreover, a number of technicalities and "well-known" facts are scattered in a vast literature as folklore, sometimes with proofs available only in the complex analytic category. This book is an attempt to fill such a gap, at least partially. More precisely, it aims at giving an account with complete proofs of the results and techniques which are needed to understand the local deformation theory of algebraic schemes over an algebraically closed field, thus providing the tools needed, for example, in the local study of Hilbert schemes and moduli problems. The existing monographs, like [14], [93], [105], [109], [124], [163], [175], [176], [184], all aim at goals different from the above.

For these reasons my approach has been to work exclusively in the category of locally noetherian schemes over a fixed algebraically closed field **k**, to avoid switching back and forth between the algebraic and the analytic category. I tried to make the text self-contained as much as possible, but without forgetting that all the technical ideas and prerequisites can be found in [3] and [2]: therefore the reader is advised to keep a copy of them to hand while reading this text. In any case a good familiarity with [84] and with a standard text in commutative algebra like [48] or [127] will be generally sufficient; the classical [167] and [190] will be also useful. A good acquaintance with homological algebra is assumed throughout.

One of the difficulties of writing about this subject is that it needs a great number of technical results, which make it hard to maintain a proper balance between generality and understandability. In order to overcome this problem I tried to keep the technicalities to a minimum, and I introduced the main deformation problems in an elementary fashion in Chapter 1; they are then reconsidered as functors of Artin rings in Chapter 2, where the main results of the theory are proved. The first two chapters therefore give a self-contained treatment of formal deformation theory via the "classical" approach; cotangent complexes and functors are not introduced, nor the method of differential graded Lie algebras. Another chapter treats in more detail the most important deformation functors, with the single exception of vector bundles; this was motivated by reasons of space and because good monographs on the subject are already available (e.g. [90], [118], [59]). Although they are not the central issue of the book, I considered it necessary to include a chapter on Hilbert schemes and Quot schemes, since it would be impossible to give meaningful examples and applications without them, and because of the lack of an appropriate reference. Deformation theory is closely tied with classical algebraic geometry because some of the issues which had remained controversial and unclear in the old language have found a natural explanation using the methods discussed here. I have included a section on plane curves which gives a good illustration of this point.

Unfortunately, important topics and results have been omitted because of lack of space, energy and competence. In particular, I did not include the construction of any global moduli spaces/stacks, which would have taken me too far from the main theme.

The book is organized in the following way. Chapter 1 starts with a concise treatment of algebra extensions which are fundamental in deformation theory. It then discusses locally trivial infinitesimal deformations of algebraic schemes. Chapter 2 deals with "functors of Artin rings", the abstract tool for the study of formal deformation theory. The main result of this theory is Schlessinger's theorem. A section on obstruction theory, an elementary but crucial technical point, is included. We discuss the relation between formal and algebraic deformations and the algebraization problem. This part is not entirely self-contained since Artin's algebraization theorem is not proved in general and the approximation theorem is only stated. The last section explains the role of automorphisms and the related notion of "isotriviality". Chapter 3 is an introduction to the most important deformation problems. By applying Schlessinger's theorem to them, we derive the existence of formal (semi)universal deformations. Many examples are discussed in detail so that all the basic principles of deformation theory become visible. This chapter can be used as a reference for several standard facts of deformation theory, and it can be also helpful in supplementing the study of the more abstract Chapter 2. Chapter 4 is devoted to the construction and general properties of Hilbert schemes, Quot schemes and their variants, the "flag Hilbert schemes". It ends with a section on plane curves, where the main properties of Severi varieties are discussed. My approach to the proof of existence of nodal curves with any number of nodes uses multiple point schemes and is apparently new.

In the Appendices I have collected several topics which are well known and standard but I felt it would be convenient for the reader to have them available here.

Acknowledgements. Firstly, I would like to express my deepest gratitude to D. Lieberman, who introduced me to the study of deformations of complex manifolds a long time ago. More recently, R. Hartshorne, after a careful reading of a previous draft of this book, made a number of comments and suggestions which significantly contributed in improving it. I warmly thank him for his generous help. For many extremely useful remarks on another draft of the book I am also indebted to M. Brion. I gratefully acknowledge comments and suggestions from other colleagues and students, in particular: L. Badescu, I. Bauer, A. Bruno, M. Gonzalez, A. Lopez, M. Manetti, A. Molina Rojas, D. Tossici, A. Verra, A. Vistoli.

I would also like to thank L. Caporaso, F. Catanese, C. Ciliberto, L. Ein, D. Laksov and H. Lange for encouragement and support which have been of great help.

Contents

Terminology and notation

All rings will be commutative with 1. A ring homomorphism $A \to B$ is called *essentially of finite type* (e.f.t.) if B is a localization of an A-algebra of finite type. We will also say that B is e.f.t. over A.

We will always denote by $\mathbf{k}$ a fixed algebraically closed field. All schemes will be assumed to be defined over $\mathbf{k}$, locally noetherian and separated, and all algebraic sheaves will be quasi-coherent unless otherwise specified. If X and Y are schemes we will write $X \times Y$ instead of $X \times_{\mathbf{k}} Y$. If S is a scheme and $s \in S$ we denote by $\mathbf{k}(s) = \mathcal{O}_{S,s}/m_{S,s}$ the residue field of S at s.

As is customary, various categories will be denoted by indicating their objects within parentheses when it will be clear what the morphisms in the category are. For instance (sets), (A-modules), etc. The class of objects of a category $\mathcal{C}$ will be denoted by ob($\mathcal{C}$). The dual of a category $\mathcal{C}$ will be denoted by $\mathcal{C}^\circ$. Given categories $\mathcal{C}$ and $\mathcal{D}$, a contravariant functor F from $\mathcal{C}$ to $\mathcal{D}$ will be always denoted as a covariant functor $F : \mathcal{C}^\circ \to \mathcal{D}$.

We will consider the following categories of $\mathbf{k}$-algebras:

$\mathcal{A}$ = the category of local artinian $\mathbf{k}$-algebras with residue field $\mathbf{k}$
$\hat{\mathcal{A}}$ = the category of complete local noetherian $\mathbf{k}$-algebras with residue field $\mathbf{k}$
$\mathcal{A}^*$ = the category of local noetherian $\mathbf{k}$-algebras with residue field $\mathbf{k}$
($\mathbf{k}$-algebras) = the category of noetherian $\mathbf{k}$-algebras

Morphisms are unitary $\mathbf{k}$-homomorphisms, which are local in $\mathcal{A}$, $\hat{\mathcal{A}}$ and $\mathcal{A}^*$. For a given Λ in ob($\mathcal{A}^*$) we will consider the following:

$\mathcal{A}_\Lambda$ = the category of local artinian Λ-algebras with residue field $\mathbf{k}$
$\mathcal{A}^*_\Lambda$ = the category of local noetherian Λ-algebras with residue field $\mathbf{k}$

They are subcategories of $\mathcal{A}$ and $\mathcal{A}^*$ respectively. If Λ is in $\mathrm{ob}(\hat{\mathcal{A}})$ then we will let

$$\hat{\mathcal{A}}_\Lambda = \text{the category of complete local noetherian } \Lambda\text{-algebras with residue field } \mathbf{k}$$

which is a subcategory of $\hat{\mathcal{A}}$. Moreover, we will set:

$$(\text{schemes}) = \text{the category of schemes}$$

(i.e. of locally noetherian separated **k**-schemes) and

$$(\text{algschemes}) = \text{the category of algebraic schemes}$$

For a given scheme Z we set

$$(\text{schemes}/Z) = \text{the category of } Z\text{-schemes}$$
$$(\text{algschemes}/Z) = \text{the category of algebraic } Z\text{-schemes}$$

$h^i(X, \mathcal{F})$ denotes $\dim[H^i(X, \mathcal{F})]$ where $\mathcal{F}$ is a coherent sheaf on the complete scheme X. When no confusion is possible we will sometimes write $H^i(\mathcal{F})$ and $h^i(\mathcal{F})$ instead of $H^i(X, \mathcal{F})$ and $h^i(X, \mathcal{F})$ respectively.

$\coprod_i X_i$ denotes the disjoint union of the schemes X_i.

If E is a vector space or a locally free sheaf we will always denote its dual by $E^\vee$. If V is a **k**-vector space we will denote by $\mathbb{P}(V)$ the projective space $\mathrm{Proj}(\mathrm{Sym}(V^\vee))$ (where $\mathrm{Sym}(-)$ is the symmetric algebra of $-$): thus the closed points of $\mathbb{P}(V)$ are the one-dimensional subspaces of V. Similarly, if E is a locally free sheaf on an algebraic scheme S, *the projective bundle associated to* E will be defined as

$$\mathbb{P}(E) = \mathrm{Proj}(\mathrm{Sym}(E^\vee))$$

Note that this definition is dual to the one given in [84], p. 162.

For all definitions not explicitly given we will refer to [84].

Introduction

La méthode générale consiste toujours à faire des constructions formelles, ce qui consiste essentiellement à faire de la géométrie algébrique sur un anneau artinien, et à en tirer des conclusions de nature "algébrique" en utilisant les trois théorèmes fondamentaux (Grothendieck [71], p. 11).

Deformation theory is a formalization of the Kodaira–Nirenberg–Spencer–Kuranishi (KNSK) approach to the study of small deformations of complex manifolds. Its main ideas are clearly outlined in the series of Bourbaki seminar expositions by Grothendieck which go under the name of "Fondements de la Géométrie Algébrique" [2]; in particular they are explained in detail in [72] (see especially page 17), while the technical foundations are laid in [71]. The quotation at the top of this page gives a concise description of the method employed.

The first step of this formalization consists in studying infinitesimal deformations, and this is accomplished via the notion of "functor of Artin rings"; the study of such functors leads to the construction of "formal deformations". This method enhances the analogies between the analytic and the algebraic cases, and at the same time hides some delicate phenomena typical of the algebraic geometrical world. These phenomena become visible when one tries to pass from formal to algebraic deformations. The techniques of deformation theory have a variety of applications which make them an extremely useful tool, especially in understanding the local structure of schemes defined by geometrical conditions or by functorial constructions.

In this introduction we shall explain in outline the logical structure of deformation theory; for this purpose we will start by outlining the KNSK theory of small deformations of compact complex manifolds.

Given a compact complex manifold X, a *family of deformations* of X is a commutative diagram of holomorphic maps between complex manifolds

$$\begin{array}{cccc} & X & \subset & \mathcal{X} \\ \xi: & \downarrow & & \downarrow \pi \\ & \star & \xrightarrow{t_o} & B \end{array}$$

with π proper and smooth (i.e. with everywhere surjective differential), B connected and where $\star$ denotes the singleton space. We denote by $\mathcal{X}_t$ the fibre $\pi^{-1}(t)$, $t \in B$. It is a standard fact that, locally on B, $\mathcal{X}$ is differentiably a product so that π can be viewed locally as a family of complex structures on the differentiable manifold X_{diff}.

The family ξ is *trivial* at t_o if there is a neighbourhood $U \subset B$ of t_o such that we have $\pi^{-1}(U) \cong X \times U$ analytically.

Kodaira and Spencer started by defining, for every tangent vector $\frac{\partial}{\partial t} \in T_{t_o}B$, *the derivative of the family* π *along* $\frac{\partial}{\partial t}$ as an element

$$\frac{\partial \mathcal{X}_t}{\partial t} \in H^1(X, T_X)$$

thus giving a linear map

$$\kappa : T_{t_o}B \to H^1(X, T_X)$$

called the *Kodaira–Spencer map* of the family π. They showed that if π is trivial at t_o then $\kappa(\frac{\partial}{\partial t}) = 0$ for all $\frac{\partial}{\partial t} \in T_{t_o}B$. Then they investigated the problem of classifying all small deformations of X, by constructing a "complete family" of deformations of X. A family ξ as above is called *complete* if for every other family of deformations of X:

$$\begin{array}{cccc} & X & \subset & \mathcal{Y} \\ \eta : & \downarrow & & \downarrow p \\ & \star & \xrightarrow{m_o} & M \end{array}$$

there is an open neighbourhood $V \subset M$ and a commutative diagram

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & & \searrow & \\ p^{-1}(V) & & \to & & \mathcal{X} \\ \downarrow & & & & \downarrow \\ V & & \to & & B \end{array}$$

inducing an isomorphism $p^{-1}(V) \cong V \times_B \mathcal{X}$. The family is called *universal* if it is complete and moreover, the morphism $V \to B$ is unique locally around m_o for each family η as above. Kodaira and Spencer proved that if κ is surjective then the family ξ is complete. The following existence result was then proved:

Theorem 0.0.1 (Kodaira–Nirenberg–Spencer [106]). *If* $H^2(X, T_X) = 0$ *then there exists a complete family of deformations of* X *whose Kodaira–Spencer map is an isomorphism. If, moreover,* $H^0(X, T_X) = 0$ *then such complete family is universal.*

Later Kuranishi [111] generalized this result by showing that a complete family of deformations of X such that κ is an isomorphism exists without assumptions on $H^2(X, T_X)$ provided the base B is allowed to be an analytic space.

We want to rephrase everything algebraically as far as possible. Let's fix an algebraically closed field $\mathbf{k}$ and consider an algebraic $\mathbf{k}$-scheme X. A *local deformation*, or a *local family of deformations*, of X is a cartesian diagram

$$\begin{array}{cccc} & X & \to & \mathcal{X} \\ \xi : & \downarrow & & \downarrow \pi \\ & \mathrm{Spec}(\mathbf{k}) & \subset & S \end{array}$$

where π is a flat morphism, $S = \mathrm{Spec}(A)$ where A is a local $\mathbf{k}$-algebra with residue field $\mathbf{k}$, and X is identified with the fibre over the closed point. If X is nonsingular and/or projective we will require π to be smooth and/or projective. We say that ξ is *a deformation over* Spec(A) or over A. If in particular A is an artinian local $\mathbf{k}$-algebra then we speak of an *infinitesimal deformation*.

The notion of local family has the fundamental property of being *funtorial*. Given two deformations of X:

$$\xi : \begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(\mathbf{k}) & \subset & \mathrm{Spec}(A) \end{array} \quad \text{and} \quad \eta : \begin{array}{ccc} X & \to & \mathcal{Y} \\ \downarrow & & \downarrow \rho \\ \mathrm{Spec}(\mathbf{k}) & \subset & \mathrm{Spec}(A) \end{array}$$

parametrized by the same Spec(A), an isomorphism $\xi \cong \eta$ is defined to be a morphism $f : \mathcal{X} \to \mathcal{Y}$ of schemes over Spec(A) inducing the identity on the closed fibre, i.e. such that the following diagram

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & & \searrow & \\ \mathcal{X} & & \xrightarrow{f} & & \mathcal{Y} \\ & \searrow & & \swarrow & \\ & & \mathrm{Spec}(A) & & \end{array}$$

is commutative. Consider the category

$$\mathcal{A}^* = (\text{noetherian local } \mathbf{k}\text{-algebras with residue field } \mathbf{k})$$

and its full subcategory

$$\mathcal{A} = (\text{artinian local } \mathbf{k}\text{-algebras with residue field } \mathbf{k})$$

One defines a covariant functor

$$\mathrm{Def}_X : \mathcal{A}^* \to (\text{sets})$$

by

$$\mathrm{Def}_X(A) = \{\text{local deformations of } X \text{ over Spec(A)}\}/(\text{isomorphism})$$

This is *the functor of local deformations* of X; its restriction to $\mathcal{A}$ is *the functor of infinitesimal deformations* of X. One may now ask whether Def_X is representable, namely if there is a noetherian local $\mathbf{k}$-algebra $\mathcal{O}$ and a local deformation

$$\upsilon : \begin{array}{ccc} X & \to & \mathcal{X}^\circ \\ \downarrow & & \downarrow p \\ \mathrm{Spec}(\mathbf{k}) & \subset & \mathrm{Spec}(\mathcal{O}) \end{array}$$

which is universal, i.e. such that any other local deformation ξ is obtained by pulling back υ under a unique Spec$(A) \to$ Spec$(\mathcal{O})$.

The approach of Grothendieck to this problem was to formalize the method of Kodaira and Spencer, which consists in a formal construction followed by a proof of convergence. In the search for the universal deformation υ the formal construction corresponds to the construction of the sequence of its restrictions to the truncations $\mathrm{Spec}(\mathcal{O}/m_{\mathcal{O}}^{n+1})$:

$$u_n : \begin{array}{ccc} X & \to & \mathcal{X}_n^{\circ} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \to & \mathrm{Spec}(\mathcal{O}/m_{\mathcal{O}}^{n+1}) \end{array} \qquad n \geq 0$$

These are infinitesimal deformations of X because the rings $\mathcal{O}/m_{\mathcal{O}}^{n+1}$ are in $\mathcal{A}$. The sequence $\hat{u} = \{u_n\}$ can be considered as a formal approximation of υ. It is a special case of a formal deformation: more precisely, a *formal deformation* of X is given by a complete local $\mathbf{k}$-algebra R with residue field $\mathbf{k}$ and by a sequence of infinitesimal deformations

$$\xi_n : \begin{array}{ccc} X & \to & \mathcal{X}_n \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \to & \mathrm{Spec}(R/m_R^{n+1}) \end{array} \qquad n \geq 0$$

such that $\xi_n \mapsto \xi_{n-1}$ under the truncation $R/m_R^{n+1} \to R/m_R^n$. In our case $R = \hat{\mathcal{O}}$. The goal of the formal step in deformation theory is the construction of $\hat{u}$ for a given X, i.e. of a formal deformation having a suitable universal property which is inherited from the corresponding property of υ, and which we do not need to specify now.

Observe that in trying to perform the formal step we will at best succeed in describing $\hat{\mathcal{O}}$ and not $\mathcal{O}$. Since a formal deformation consists of infinitesimal deformations, for the construction of $\hat{u}$ we will only need to work with the covariant functor

$$\mathrm{Def}_X : \mathcal{A} \to (\mathrm{sets})$$

A covariant functor $F : \mathcal{A} \to (\mathrm{sets})$ is called a *functor of Artin rings*. To every complete local $\mathbf{k}$-algebra R we can associate a functor of Artin rings h_R by

$$h_R(A) = \mathrm{Hom}_{\mathcal{A}}(R, A)$$

A functor of this form is called *prorepresentable*. By categorical general nonsense one shows that a formal deformation $\hat{\xi}$ defines a morphism of functors (a natural transformation) $h_R \to \mathrm{Def}_X$ and that this morphism is an isomorphism precisely when $\hat{\xi}$ is universal. Therefore we see that the search for $\hat{u}$ is a problem of prorepresentability of Def_X. More generally, to every local deformation problem there corresponds a functor of Artin rings F analogous to Def_X; the task of constructing a formal universal deformation for the given problem consists in showing that F is prorepresentable, producing the ring R prorepresenting F and the formal universal deformation defining the isomorphism $h_R \to F$. This is the scheme of approach to the formal part of every local deformation problem as it was outlined by Grothendieck. What one needs is to find criteria for the prorepresentability of a functor of Artin rings; we will also need to consider properties weaker than prorepresentability *(semiuniversality)* satisfied by more general classes of functors coming from interesting deformation theoretic problems. Necessary and sufficient conditions of prorepresentability and of semiuniversality are given by Schlessinger's theorem.

After having solved the problem of existence of a formal universal (or semiuniversal) deformation by means of necessary and sufficient conditions for its existence one still has to decide whether $\mathcal{O}$ and v exist and to find them. To pass from $\hat{\mathcal{O}}$ to $\mathcal{O}$ is the analogue of the convergence step in the Kodaira–Spencer theory, and it is a very difficult problem, the *algebraization problem*. Under reasonably general assumptions one shows that there exists a deformation v over an *algebraic local ring* (i.e. the henselization of a local **k**-algebra essentially of finite type) which does not quite represent the functor Def_X but at least has a universal (or semiuniversal) associated formal deformation. The further property of representing Def_X is not in general satisfied by $(\mathcal{O}, v)$, being related to the existence of nontrivial automorphisms of X. This part of the theory is largely due to the work of M. Artin, and based on the notions of *effectivity* of a formal deformation and of *local finite presentation* of a functor, already introduced by Grothendieck. The main technical tool is Artin's approximation theorem.