

Universitext

Ioannis Emmanouil

Idempotent Matrices over Complex Group Algebras

 Springer

Professor Ioannis Emmanouil

Department of Mathematics

University of Athens

157 84 Athens

Greece

E-mail: emmanoui@math.uoa.gr

Mathematics Subject Classification: 16S34, 18G, 19A31, 19D55, 20C07, 20J05, 46L10, 46L80

Library of Congress Control Number: 2005930320

ISBN-10 3-540-27990-3 Springer Berlin Heidelberg New York

ISBN-13 978-3-540-27990-7 Springer Berlin Heidelberg New York

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer. Violations are liable for prosecution under the German Copyright Law.

Springer is a part of Springer Science+Business Media

springeronline.com

© Springer-Verlag Berlin Heidelberg 2006

Printed in The Netherlands

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Typesetting: by the authors and TechBooks using a Springer L^AT_EX macro package

Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper SPIN: 11529439 41/TechBooks 5 4 3 2 1 0

στους γονείς μου

Preface

The study of idempotent elements in group algebras (or, more generally, the study of classes in the K-theory of such algebras) originates from geometric and analytic considerations. For example, C.T.C. Wall [72] has shown that the problem of deciding whether a finitely dominated space with fundamental group π is homotopy equivalent to a finite CW-complex leads naturally to the study of a certain class in the reduced K-theory $\tilde{K}_0(\mathbf{Z}\pi)$ of the group ring $\mathbf{Z}\pi$. As another example, consider a discrete group G which acts freely, properly discontinuously, cocompactly and isometrically on a Riemannian manifold. Then, following A. Connes and H. Moscovici [16], the index of an invariant 0th-order elliptic pseudo-differential operator is defined as an element in the K_0 -group of the reduced group C^* -algebra C_r^*G .

The idempotent conjecture (also known as the generalized Kadison conjecture) asserts that the reduced group C^* -algebra C_r^*G of a discrete torsion-free group G has no idempotents $\neq 0, 1$; this claim is known to be a consequence of a far-reaching conjecture of P. Baum and A. Connes [6]. Alternatively, one may approach the idempotent conjecture as an assertion about the connectedness of a *non-commutative space*; if G is a discrete torsion-free abelian group then C_r^*G is the algebra of continuous complex-valued functions on the dual group \widehat{G} , which is itself a compact and connected topological space. Even though the complex group algebra of a group G is a much simpler object than the corresponding reduced group C^* -algebra, the idempotent conjecture for $\mathbf{C}G$ remains still an unproved claim when G is an arbitrary torsion-free group. The latter problem has attracted the attention of ring theorists since the middle of the 20th century.

On the other hand, reformulating a theorem of R. Swan [70] about projective modules over integral group rings of finite groups in terms of the Hattori-Stallings rank, H. Bass stated in [4,5] a conjecture about the trace of idempotent matrices with entries in the group algebra of a group with coefficients in a suitable subring of the field \mathbf{C} of complex numbers. An immediate consequence of the validity of the conjecture is the equality of various Euler characteristics

that can be defined for groups. Furthermore, as shown by B. Eckmann [20], Bass' conjecture is related to the freeness of certain induced finitely generated projective modules over the von Neumann algebra of the group.

This book provides an introduction to the study of these problems for graduate students and researchers who are not necessarily experts in the field. Our aim is to show the unified character of the conjectures mentioned above and present the basic elements of an area of research that has recently experienced a revival, in view of its close relationship with deep geometric problems. At the same time, we hope that this book will become a valuable aid to the experts as well, as it collects and presents in a systematic way basic techniques and important results that have been obtained during the past few decades.

The pace of the book is suitable for independent study and the level of the presentation not very demanding, assuming only familiarity with the techniques of Algebra and Analysis that are usually covered during the first year of graduate studies. Moreover, in order to facilitate the reader, we have decided to include a few Appendices that detail some of the tools used in the main text. On the other hand, we have restrained ourselves from using some of the more advanced techniques that may be employed in the study of these problems, such as refined tools from K-theory.

In the first chapter, we fix the notation used in the rest of the book and properly formulate the Bass' and idempotent conjectures. As a warm-up, we prove the idempotent conjecture for torsion-free ordered groups and introduce the Strebel-Strojnowski class, providing some additional examples of groups satisfying the idempotent conjecture.

In Chap. 2, we present the simplest examples of groups that satisfy Bass' conjecture, namely the abelian groups and the finite ones. We put the conjecture in a geometric perspective, by relating it, in the abelian case, to the connectedness of the prime spectrum of the group algebra. Using some basic representation theory, we establish the equivalence between Bass' conjecture for finite groups and Swan's theorem on integral representations, which served itself as the primary motivation for H. Bass to formulate the conjecture.

In Chap. 3, we study idempotent matrices with entries in complex group algebras by reduction to positive characteristic. This technique was pioneered by A. Zaleskii [75], in order to complement a result of I. Kaplansky [38] on the positivity of the canonical trace. Using the action of the Frobenius operator in the positive characteristic case and then lifting the result to \mathbf{C} , we prove two theorems of H. Bass and P. Linnell describing some properties of the support of the Hattori-Stallings rank of an idempotent matrix.

In Chap. 4, we present another method that may be used in the study of the idempotent conjectures, which is of homological nature. We define cyclic homology and relate it to the K-theory and the Hattori-Stallings rank. The nilpotency of Connes' periodicity operator in the cyclic homology of group algebras suggests the definition of a class \mathcal{C} , which provides us with many interesting examples of groups that satisfy the idempotent conjectures.

In the last chapter, we study idempotent matrices with entries in the reduced group C^* -algebra of a discrete group and prove the integrality of the canonical trace, in the cases of abelian and free groups. In the abelian case, this follows from the connectedness of the dual group, whereas the free group case is taken care of by considering a free action of the group on a tree. We construct the center-valued trace on the von Neumann algebra of a group from scratch (i.e. without appealing to the general theory of finite algebras) and study its importance in K-theory. In particular, we prove the result of B. Eckmann on the freeness of induced finitely generated projective modules over group von Neumann algebras.

The five Appendices at the end of the book summarize the results from Algebra, Number Theory and Analysis that are needed in the main text.

At this point, I would like to acknowledge the intellectual debt owed to the mathematicians whose work and ideas build up this book; in particular, to H. Bass, A. Connes, B. Eckmann, I. Kaplansky, P. Linnell and A. Zaleskii.

Athens, Greece
June, 2005

Ioannis Emmanouil

Contents

1	Introduction	1
1.1	Preliminaries	1
1.1.1	Basic Notions from Algebra	1
1.1.2	Basic Notions from Analysis	8
1.1.3	The K_0 -group of a Ring	15
1.1.4	Traces and the K_0 -group	23
1.2	The Idempotent Conjectures	30
1.2.1	The Hattori-Stallings Rank on $K_0(kG)$	30
1.2.2	Idempotents in \mathbf{CG}	34
1.2.3	Some First Examples of Groups that Satisfy the Idempotent Conjecture	37
1.3	Exercises	42
2	Motivating Examples	49
2.1	The Case of Abelian Groups	49
2.1.1	The Geometric Rank Function	50
2.1.2	K-theory and the Geometric Rank	52
2.1.3	The Connectedness of $\text{Spec } kG$	59
2.2	The Case of Finite Groups	61
2.2.1	The Transfer Homomorphism	61
2.2.2	Subgroups of Finite Index	63
2.2.3	Swan's Theorem	65
2.3	Exercises	68
3	Reduction to Positive Characteristic	73
3.1	The Rationality of the Canonical Trace	73
3.1.1	Coefficient Fields of Positive Characteristic	74
3.1.2	Lifting to the Field of Algebraic Numbers	77
3.1.3	The Kaplansky Positivity Theorem	80
3.1.4	Idempotent Matrices with Entries in the Complex Group Algebra	85

3.2	The Support of the Hattori-Stallings Rank	87
3.2.1	Iterates of the Frobenius Operator	87
3.2.2	The Main Results	91
3.2.3	An Application: the Case of Solvable Groups	100
3.3	Exercises	106
4	A Homological Approach	111
4.1	Cyclic Homology of Algebras	111
4.1.1	Basic Definitions and Results	112
4.1.2	The Relation to K-theory	125
4.1.3	The Cyclic Homology of Group Algebras	129
4.2	The Nilpotency of Connes' Operator	145
4.2.1	Idempotent Conjectures and the Nilpotency of S	145
4.2.2	Closure Properties	149
4.3	Exercises	155
5	Completions of CG	159
5.1	The Integrality of the Trace Conjecture	159
5.1.1	Formulation of the Conjecture	160
5.1.2	The Case of an Abelian Group	161
5.1.3	The Case of a Free Group	170
5.2	Induced Modules over $\mathcal{N}G$	179
5.2.1	The Center-Valued Trace on $\mathcal{N}G$	180
5.2.2	Matrices with Entries in $\mathcal{N}G$	195
5.3	Exercises	202
A	Tools from Commutative Algebra	207
A.1	Localization and Local Rings	207
A.2	Integral Dependence	213
A.3	Noether Normalization	217
A.4	The Krull Intersection Theorem	222
A.5	Exercises	225
B	Discrete Ring-Valued Integrals	227
B.1	Discrete Group-Valued Integrals	227
B.2	Idempotent-Valued Premeasures	230
B.3	Exercises	232
C	Frobenius' Density Theorem	235
C.1	The Density Theorem	235
C.2	Exercises	238

D Homological Techniques 239

 D.1 Complexes and Homology 239

 D.1.1 Chain Complexes 239

 D.1.2 Double Complexes 240

 D.1.3 Tor and Ext 242

 D.2 Group Homology and Cohomology 244

 D.2.1 Basic Definitions 244

 D.2.2 H^2 and Extensions 247

 D.2.3 Products 249

 D.2.4 Duality 253

 D.2.5 The (co-)homology of an Extension 255

 D.3 Exercises 257

E Comparison of Projections 263

 E.1 Equivalence and Weak Ordering 263

 E.2 Exercises 269

References 271

Index 275