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Idempotent Matrices over Complex Group Algebras

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Preface

The study of idempotent elements in group algebras (or, more generally, the study of classes in the K-theory of such algebras) originates from geometric and analytic considerations. For example, C.T.C. Wall [72] has shown that the problem of deciding whether a finitely dominated space with fundamental group π is homotopy equivalent to a finite CW-complex leads naturally to the study of a certain class in the reduced K-theory $\widetilde{K}_0(\mathbf{Z}\pi)$ of the group ring $\mathbf{Z}\pi$. As another example, consider a discrete group G which acts freely, properly discontinuously, cocompactly and isometrically on a Riemannian manifold. Then, following A. Connes and H. Moscovici [16], the index of an invariant 0th-order elliptic pseudo-differential operator is defined as an element in the K_0-group of the reduced group C^*-algebra $C^*_r G$.

The idempotent conjecture (also known as the generalized Kadison conjecture) asserts that the reduced group C^*-algebra $C^*_r G$ of a discrete torsion-free group G has no idempotents $\neq 0, 1$; this claim is known to be a consequence of a far-reaching conjecture of P. Baum and A. Connes [6]. Alternatively, one may approach the idempotent conjecture as an assertion about the connectedness of a *non-commutative space*; if G is a discrete torsion-free abelian group then $C^*_r G$ is the algebra of continuous complex-valued functions on the dual group $\widehat{G}$, which is itself a compact and connected topological space. Even though the complex group algebra of a group G is a much simpler object than the corresponding reduced group C^*-algebra, the idempotent conjecture for $\mathbf{C}G$ remains still an unproved claim when G is an arbitrary torsion-free group. The latter problem has attracted the attention of ring theorists since the middle of the 20th century.

On the other hand, reformulating a theorem of R. Swan [70] about projective modules over integral group rings of finite groups in terms of the Hattori-Stallings rank, H. Bass stated in [4,5] a conjecture about the trace of idempotent matrices with entries in the group algebra of a group with coefficients in a suitable subring of the field $\mathbf{C}$ of complex numbers. An immediate consequence of the validity of the conjecture is the equality of various Euler characteristics that can be defined for groups. Furthermore, as shown by B. Eckmann [20], Bass' conjecture is related to the freeness of certain induced finitely generated projective modules over the von Neumann algebra of the group.

This book provides an introduction to the study of these problems for graduate students and researchers who are not necessarily experts in the field. Our aim is to show the unified character of the conjectures mentioned above and present the basic elements of an area of research that has recently experienced a revival, in view of its close relationship with deep geometric problems. At the same time, we hope that this book will become a valuable aid to the experts as well, as it collects and presents in a systematic way basic techniques and important results that have been obtained during the past few decades.

The pace of the book is suitable for independent study and the level of the presentation not very demanding, assuming only familiarity with the techniques of Algebra and Analysis that are usually covered during the first year of graduate studies. Moreover, in order to facilitate the reader, we have decided to include a few Appendices that detail some of the tools used in the main text. On the other hand, we have restrained ourselves from using some of the more advanced techniques that may be employed in the study of these problems, such as refined tools from K-theory.

In the first chapter, we fix the notation used in the rest of the book and properly formulate the Bass' and idempotent conjectures. As a warm-up, we prove the idempotent conjecture for torsion-free ordered groups and introduce the Strebel-Strojnowski class, providing some additional examples of groups satisfying the idempotent conjecture.

In Chap. 2, we present the simplest examples of groups that satisfy Bass' conjecture, namely the abelian groups and the finite ones. We put the conjecture in a geometric perspective, by relating it, in the abelian case, to the connectedness of the prime spectrum of the group algebra. Using some basic representation theory, we establish the equivalence between Bass' conjecture for finite groups and Swan's theorem on integral representations, which served itself as the primary motivation for H. Bass to formulate the conjecture.

In Chap. 3, we study idempotent matrices with entries in complex group algebras by reduction to positive characteristic. This technique was pioneered by A. Zaleskii [75], in order to complement a result of I. Kaplansky [38] on the positivity of the canonical trace. Using the action of the Frobenius operator in the positive characteristic case and then lifting the result to $\mathbf{C}$, we prove two theorems of H. Bass and P. Linnell describing some properties of the support of the Hattori-Stallings rank of an idempotent matrix.

In Chap. 4, we present another method that may be used in the study of the idempotent conjectures, which is of homological nature. We define cyclic homology and relate it to the K-theory and the Hattori-Stallings rank. The nilpotency of Connes' periodicity operator in the cyclic homology of group algebras suggests the definition of a class $\mathcal{C}$, which provides us with many interesting examples of groups that satisfy the idempotent conjectures.

In the last chapter, we study idempotent matrices with entries in the reduced group C^*-algebra of a discrete group and prove the integrality of the canonical trace, in the cases of abelian and free groups. In the abelian case, this follows from the connectedness of the dual group, whereas the free group case is taken care of by considering a free action of the group on a tree. We construct the center-valued trace on the von Neumann algebra of a group from scratch (i.e. without appealing to the general theory of finite algebras) and study its importance in K-theory. In particular, we prove the result of B. Eckmann on the freeness of induced finitely generated projective modules over group von Neumann algebras.

The five Appendices at the end of the book summarize the results from Algebra, Number Theory and Analysis that are needed in the main text.

At this point, I would like to acknowledge the intellectual debt owed to the mathematicians whose work and ideas build up this book; in particular, to H. Bass, A. Connes, B. Eckmann, I. Kaplansky, P. Linnell and A. Zaleskii.

Athens, Greece *Ioannis Emmanouil*
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Contents