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Combinatorics of Coxeter Groups

With 81 Illustrations

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To Annamaria and Christine

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Foreword

Coxeter groups arise in a multitude of ways in several areas of mathematics. They are studied in algebra, geometry, and combinatorics, and certain aspects are of importance also in other fields of mathematics. The theory of Coxeter groups has been exposited from algebraic and geometric points of view in several places, also in book form. The purpose of this work is to present its core combinatorial aspects.

By "combinatorics of Coxeter groups" we have in mind the mathematics that has to do with reduced expressions, partial order of group elements, enumeration, associated graphs and combinatorial cell complexes, and connections with combinatorial representation theory. There are some other topics that could also be included under this general heading (e.g., combinatorial properties of reflection hyperplane arrangements on the geometric side and deeper connections with root systems and representation theory on the algebraic side). However, with the stated aim, there is already more than plenty of material to fill one volume, so with this "disclaimer" we limit ourselves to the chosen core topics.

It is often the case that phenomena of Coxeter groups can be understood in several ways, using either an algebraic, a geometric, or a combinatorial approach. The interplay between these aspects provides the theory with much of its richness and depth. When alternate approaches are possible, we usually choose a combinatorial one, since it is our task to tell this side of the story. For a more complete understanding of the subject, the reader is urged to study also its algebraic and geometric aspects. The notes at the end of each chapter provide references and hints for further study.

The book is divided into two parts. The first part, comprising Chapters 1 – 4, gives a self-contained introduction to combinatorial Coxeter group theory. We treat the combinatorics of reduced decompositions, Bruhat order, weak order, and some aspects of root systems. The second part consists of four independent chapters dealing with certain more advanced topics. In Chapters 5 – 7, some external references are necessary, but we have tried to minimize reliance on other sources. Chapter 8, which is elementary, discusses permutation representations of the most important finite and affine Coxeter groups.

Exercises are provided to all chapters — both easier exercises, meant to test understanding of the material, and more difficult ones representing results from the research literature. Open problems are marked with an asterisk. Thus, the book is meant to have a dual character as both graduate textbook (particularly Part I) and as research monograph (particularly Part II).

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Stockholm and Rome, September 2004

Anders Björner and Francesco Brenti

Notation

We collect here some notation that is adhered to throughout the book.

$\mathbb{Z}$	the integers
$\mathbb{N}$	the non-negative integers
$\mathbb{P}$	the positive integers
$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	the rational, real, and complex numbers
$[n]$	the set $\{1, 2, \ldots, n\}$ $(n \in \mathbb{N})$, in particular $[0] = \emptyset$
$[a, b]$	the set $\{n \in \mathbb{Z} : a \leq n \leq b\}$ $(a, b \in \mathbb{Z})$
$[\pm n]$	the set $[-n, n] \setminus \{0\}$
$\{a_1, \ldots, a_n\}_<$	the set $\{a_1, \ldots, a_n\}$ with total order $a_1 < \cdots < a_n$
$\lfloor a \rfloor$	the largest integer $\leq a$ $(a \in \mathbb{R})$
$\lceil a \rceil$	the smallest integer $\geq a$ $(a \in \mathbb{R})$
$\operatorname{sgn}(a)$	the sign of a real number: $\operatorname{sgn}(a) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } a > 0, \\ 0, & \text{if } a = 0, \\ -1, & \text{if } a < 0. \end{cases}$
δ_{ij} or $\delta(i, j)$	the Kronecker delta: $\delta_{ij} \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$
$\lvert A \rvert$, $\#A$, or $\operatorname{card}(A)$	the cardinality of a set A
$A \uplus B$	the union of two disjoint sets
$A \Delta B$	the symmetric difference $A \cup B \setminus (A \cap B)$
2^A	the family of all subsets of a finite set A
$\binom{A}{k}$	the family of all k-element subsets of a finite set A
A^*	the set of all words with letters from an alphabet A

Each result (theorem, corollary, proposition, or lemma) is numbered consecutively within sections. So, for example, Theorem 2.3.3 is the third result in the third section of Chapter 2 (i.e., in Section 2.3). The symbol $\square$ denotes the end of a proof or an example. A $\square$ appearing at the end of the statement of a result signifies that the result should be obvious at that stage of reading, or else that a reference to a proof is given.