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Complex Geometry

An Introduction

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Preface

Complex geometry is a highly attractive branch of modern mathematics that has witnessed many years of active and successful research and that has recently obtained new impetus from physicists' interest in questions related to mirror symmetry. Due to its interactions with various other fields (differential, algebraic, and arithmetic geometry, but also string theory and conformal field theory), it has become an area with many facets. Also, there are a number of challenging open problems which contribute to the subject's attraction. The most famous among them is the Hodge conjecture, one of the seven one-million dollar millennium problems of the Clay Mathematics Institute. So, it seems likely that this area will fascinate new generations for many years to come.

Complex geometry, as presented in this book, studies the geometry of (mostly compact) complex manifolds. A complex manifold is a differentiable manifold endowed with the additional datum of a complex structure which is much more rigid than the geometrical structures in differential geometry. Due to this rigidity, one is often able to describe the geometry of complex manifolds in very explicit terms. E.g. the important class of projective manifolds can, in principle, be described as zero sets of polynomials.

Yet, a complete classification of all compact complex manifolds is too much to be hoped for. Complex curves can be classified in some sense (involving moduli spaces etc.), but already the classification of complex surfaces is tremendously complicated and partly incomplete.

In this book we will concentrate on more restrictive types of complex manifolds for which a rather complete theory is in store and which are also relevant in the applications. A prominent example are Calabi–Yau manifolds, which play a central role in questions related to mirror symmetry. Often, interesting complex manifolds are distinguished by the presence of special Riemannian metrics. This will be one of the central themes throughout this text. The idea is to study cases where the Riemannian and complex geometry on a differentiable manifold are not totally unrelated. This inevitably leads to

Kähler manifolds, and a large part of the book is devoted to techniques suited for the investigation of this prominent type of complex manifolds.

The book is based on a two semester course taught in 2001/2002 at the university of Cologne. It assumes, besides the usual facts from the theory of holomorphic functions in one variable, the basic notions of differentiable manifolds and sheaf theory. For the convenience of the reader we have summarized those in the appendices A and B. The aim of the course was to introduce certain fundamental concepts, techniques, and results in the theory of compact complex manifolds, without being neither too basic nor too sketchy.

I tried to teach the subject in a way that would enable the students to follow recent developments in complex geometry and in particular some of the exciting aspects of the interplay between complex geometry and string theory. Thus, I hope that the book will be useful for both communities, those readers aiming at understanding and doing research in complex geometry and those using mathematics and especially complex geometry in mathematical physics.

Some of the material was intended rather as an outlook to more specialized topics, and I have added those as appendices to the corresponding chapters. They are not necessary for the understanding of the subsequent sections.

I am aware of several shortcomings of this book. As I found it difficult to teach the deeper aspects of complex analysis to third-year students, the book cannot serve as an introduction to the fascinating program initiated by Siu, Demailly, and others, that recently has lead to important results in complex and algebraic geometry. So, for the analysis I have to refer to Demailly's excellent forthcoming (?) text book [35]. I also had to leave out quite a number of important tools, like higher direct image sheaves, spectral sequences, intermediate Jacobians, and others. The hope was to create a streamlined approach to some central results and so I did not want to enter too many of the promising side-roads. Finally, although relevant examples have been included in the text as well as in the exercises, the book does not discuss in depth any difficult type of geometry, e.g. Calabi–Yau or hyperkähler manifolds. But I believe that with the book at hand, it should not be too difficult to understand more advanced texts on special complex manifolds.

Besides Demailly's book [35], there are a number of text books on complex geometry, Hodge theory, etc. The classic [59] and the more recent one by Voisin [113] are excellent sources for more advanced reading. I hope that this book may serve as a leisurely introduction to those.

In the following, we will give an idea of the content of the book. For more information, the reader may consult the introductions at the beginning of each chapter.

Chapter 1 provides the minimum of the local theory needed for the global description of complex manifolds. It may be read along with the later chapters or worked through before diving into the general theory of complex manifolds beginning with Chapter 2.

Section 1.1 shows a way from the theory of holomorphic functions of one variable to the general theory of complex functions. Eventually, it would lead to the local theory of complex spaces, but we restrict ourselves to those aspects strictly necessary for the understanding of the rest of the book. The reader interested in this attractive combination of complex analysis and commutative algebra may consult [35] or any of the classics, e.g. [57, 64].

Section 1.2 is a lesson in linear algebra and as such rather elementary. We start out with a real vector space endowed with a scalar product and the additional datum of an almost complex structure. We shall investigate what kind of structure is induced on the exterior algebra of the vector space. I tried to present the material with some care in order to make the reader feel comfortable when later, in the global context, the machinery is applied to compact Kähler manifolds.

Section 1.3 proves holomorphic versions of the Poincaré lemma and is supposed to accustom the reader to the yoga of complex differential forms on open sets of $\mathbb{C}^n$.

With **Chapter 2** the story begins. Sections 2.1 and 2.2 deal with complex manifolds and holomorphic vector bundles, both holomorphic analogues of the corresponding notions in real differential geometry. But a few striking differences between the real and the complex world will become apparent right away. The many concrete examples of complex manifolds are meant to motivate the discussion of the more advanced techniques in the subsequent chapters.

Section 2.3 illuminates the intimate relation between complex codimension one submanifolds (or, more generally, divisors) and holomorphic line bundles with their global sections. This builds the bridge to classical algebraic geometry, e.g. Veronese and Segre embedding are discussed. The section ends with a short discussion of the curve case.

Section 2.4 is devoted to the complex projective space $\mathbb{P}^n$, a universal object in complex (algebraic) geometry comparable to spheres in the real world. We describe its tangent bundle by means of the Euler sequence and certain tautological line bundles. A discussion of the Riemannian structure of $\mathbb{P}^n$ (e.g. the Fubini–Study metric) is postponed until Section 3.1.

Section 2.5 provides an example of the universal use of the projective space. It explains a complex surgery, called blow-up, which modifies a given complex manifold along a distinguished complex submanifold, replacing a point by a projective space. Apart from its importance in the birational classification of complex manifolds, blow-ups will turn out to be of use in the proof of the Kodaira embedding theorem in Section 5.2.

Section 2.6 interprets complex manifolds as differentiable manifolds together with an additional linear datum (an almost complex structure) satisfying an integrability condition. Here, the linear algebra of Section 1.2 comes in handy. The crucial Newlander–Nierenberg theorem, asserting the equivalence of the two points of view, is formulated but not proved.

Chapter 3 is devoted to (mostly compact) Kähler manifolds. The existence of a Kähler metric on a compact complex manifold has far reaching consequences for its cohomology. Behind most of the results on Kähler manifolds one finds the so-called Kähler identities, a set of commutator relations for the various differential and linear operators. They are the topic of Section 3.1.

In Section 3.2, Hodge theory for compact manifolds is used to pass from arbitrary forms to harmonic forms and eventually to cohomology classes. This immediately yields central results, like Serre duality and, in Section 3.3, Lefschetz decomposition.

Section 3.3 also explains how to determine those classes in the second cohomology $H^2(X)$ of a compact Kähler manifold X that come from holomorphic line bundles. This is the Lefschetz theorem on $(1,1)$-classes. A short introduction to the hoped for generalization to higher degree cohomology classes, i.e. the Hodge conjecture, ends this section.

There are three appendices to Chapter 3. Appendix 3.A proves the formality of compact Kähler manifolds, a result that interprets the crucial $\partial\bar{\partial}$-lemma of Section 3.2 homologically. Appendix 3.B is a first introduction to some mathematical aspects of supersymmetry. The cohomological structures encountered in the bulk of the chapter are formalized by the notion of a Hodge structure. Appendix 3.C collects a few basic notions and explains how they fit in our context.

Chapter 4 provides indispensable tools for the study of complex manifolds: connections, curvature, and Chern classes. In contrast to previous sections, we will not just study complex manifolds and their tangent bundles but broaden our perspective by considering arbitrary holomorphic vector bundles. However, we will not be in the position to undertake an indepth analysis of all fundamental questions. E.g. the question whether there exist holomorphic vector bundles besides the obvious ones on a given manifold (or holomorphic structures on a given complex vector bundle) will not be addressed. This is partially due to the limitations of the book, but also to the state of the art. Only for curves and projective surfaces the situation is fairly well understood (see [70]).

In the appendices to Chapter 4 we discuss the interplay of complex and Riemannian geometry. Appendix 4.A tries to clarify the relation between the Levi-Civita connection and the Chern connection on a Kähler manifold. The concept of holonomy, well known in classical Riemannian geometry, allows to view certain features in complex geometry from a slightly different angle. Appendix 4.B outlines some basic results about Kähler–Einstein and Hermite–Einstein metrics. Before, the hermitian structure on a holomorphic vector bundle was used as an auxiliary in order to apply Hodge theory, etc. Now, we ask whether canonical hermitian structures, satisfying certain compatibility conditions, can be found.

In order to illustrate the power of cohomological methods, we present in **Chapter 5** three central results in complex algebraic geometry. Except for the

Hirzebruch–Riemann–Roch theorem, complete proofs are given, in particular for Kodaira's vanishing and embedding theorems. The latter one determines which compact complex manifolds can be embedded into a projective space. All three results are of fundamental importance in the global theory of complex manifolds.

Chapter 6 is relevant to the reader interested in Calabi–Yau manifolds and mirror symmetry. It is meant as a first encounter with deformation theory, a notoriously difficult and technical subject. In Section 6.1 we leave aside convergence questions and show how to study deformations by a power series expansion leading to the Maurer–Cartan equation. This approach can successfully be carried out for compact Kähler manifolds with trivial canonical bundle (Calabi–Yau manifolds) due to the Tian–Todorov lemma. Section 6.2 surveys the more abstract parts of deformation theory, without going into detail. The appendix to this chapter is very much in the spirit of appendix 3.A. Here, the content of Section 6.1 is put in the homological language of Batalin–Vilkovisky algebras, a notion that has turned out to be useful in the construction of Frobenius manifolds and in the formulation of mirror symmetry.

In general, all results are proved except for assertions presented as 'theorems', indicating that they are beyond the scope of this book, and a few rather sketchy points in the various appendices to the chapters. Certain arguments, though, are relegated to the exercises, not because I wanted to leave unpleasant bits to the reader, but because sometimes it is just more rewarding performing a computation on ones own.

Acknowledgement: I learned much of the material from the two classics [8, 59] and from my teacher H. Kurke. Later, the interplay of algebraic geometry and gauge theory as well as the various mathematical aspects of mirror symmetry have formed my way of thinking about complex geometry. The style of the presentation has been influenced by stimulating discussions with D. Kaledin, R. Thomas, and many others over the last few years.

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Paris, June 2004 *Daniel Huybrechts*

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