

Stochastic Mechanics
Random Media
Signal Processing
and Image Synthesis
Mathematical Economics and Finance
Stochastic Optimization
Stochastic Control
Stochastic Models in Life Sciences

Stochastic Modelling and Applied Probability

(Formerly:
Applications of Mathematics)

36

Edited by B. Rozovskiĭ
G. Grimmett

Advisory Board D. Dawson
D. Geman
I. Karatzas
F. Kelly
Y. Le Jan
B. Øksendal
G. Papanicolaou
E. Pardoux

Marek Musiela Marek Rutkowski

Martingale Methods in Financial Modelling

Second Edition



Springer

Authors

Marek Musiela
BNP Paribas
10 Harewood Avenue
London NW1 6AA
UK
marek.musiela@bnpparibas.com

Marek Rutkowski
Technical University Warszawa
Inst. Mathematics
Pl. Politechniki 1
00-661 Warszawa
Poland
markrut@alpha.mini.pw.edu.pl

Managing Editors

B. Rozovskiĭ
Division of Applied Mathematics
Brown University
182 George St
Providence, RI 02912
USA
rozovsky@dam.brown.edu

G. Grimmett
Centre for Mathematical Sciences
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UK
g.r.grimmett@statslab.cam.ac.uk

Cover illustration: Cover pattern courtesy of Rick Durrett,
Cornell University, Ithaca, New York.

ISBN 978-3-540-20966-9
DOI 10.1007/978-3-540-26653-2

e-ISBN 978-3-540-26653-2

Stochastic Modelling and Applied Probability ISSN 0172-4568

Library of Congress Control Number: 2004114482

Mathematics Subject Classification (2000): 60Hxx, 62P05, 90A09

2nd ed. 2005. Corr. 3rd printing 2009

© 2005, 1997 Springer-Verlag Berlin Heidelberg

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer. Violations are liable to prosecution under the German Copyright Law.

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Cover design: WMXDesign GmbH, Heidelberg

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

springer.com

Preface to the Second Edition

During the seven years that elapsed between the first and second editions of the present book, considerable progress was achieved in the area of financial modelling and pricing of derivatives. Needless to say, it was our intention to incorporate into the second edition at least the most relevant and commonly accepted of these developments. Since at the same time we had the strong intention not to expand the book to an unbearable size, we decided to leave out from the first edition of this book some portions of material of lesser practical importance.

Let us stress that we have only taken out few sections that, in our opinion, were of marginal importance for the understanding of the fundamental principles of financial modelling of arbitrage valuation of derivatives. In view of the abundance of new results in the area, it would be in any case unimaginable to cover all existing approaches to pricing and hedging financial derivatives (not to mention all important results) in a single book, no matter how voluminous it were. Hence, several intensively studied areas, such as: mean-variance hedging, utility-based pricing, entropy-based approach, financial models with frictions (e.g., short-selling constraints, bid-ask spreads, transaction costs, etc.) either remain unmentioned in this text, or are presented very succinctly. Although the issue of market incompleteness is not totally neglected, it is examined primarily in the framework of models of stochastic (or uncertain) volatility. Luckily enough, the afore-mentioned approaches and results are covered exhaustively in several excellent monographs written in recent years by our distinguished colleagues, and thus it is our pleasure to be able to refer the interested reader to these texts.

Let us comment briefly on the content of the second edition and the differences with respect to the first edition.

Part I was modified to a lesser extent and thus is not very dissimilar to Part I in the first edition. However, since, as was mentioned already, some sections from the first edition were deliberately taken out, we decided for the sake of better readability to merge some chapters. Also, we included in Part I a new chapter entirely devoted to volatility risk and related modelling issues. As a consequence, the issues of hedging of plain-vanilla options and valuation of exotic options are no longer limited to the classical Black-Scholes framework with constant volatility. The theme of stochastic volatility also reappears systematically in the second part of the book.

Part II has been substantially revised and thus its new version constitutes a major improvement of the present edition with respect to the first one. We present there alternative interest rate models, and we provide the reader with an analysis of each of them, which is very much more detailed than in the first edition. Although we did not even try to appraise the efficiency of real-life implementations of each approach, we have stressed on each occasion that, when dealing with derivatives pricing models, one should always have in mind a specific practical perspective. Put another way, we advocate the opinion, put forward by many researchers, that the choice of model should be tied to observed real features of a particular sector of the financial market or even a product class. Consequently, a necessary first step in modelling is a detailed study of functioning of a given market we wish to model. The goal of this preliminary stage is to become familiar with existing liquid primary and derivative assets (together with their sometimes complex specifications), and to identify sources of risks associated with trading in these instruments.

It was our hope that by concentrating on the most pertinent and widely accepted modelling approaches, we will be able to provide the reader with a text focused on practical aspects of financial modelling, rather than theoretical ones. We leave it, of course, to the reader to assess whether we have succeeded achieving this goal to a satisfactory level.

Marek Rutkowski expresses his gratitude to Marek Musiela and the members of the Fixed Income Research and Strategies Team at BNP Paribas for their hospitality during his numerous visits to London.

Marek Rutkowski gratefully acknowledges partial support received from the Polish State Committee for Scientific Research under grant PBZ-KBN-016/P03/1999.

We would like to express our gratitude to the staff of Springer-Verlag. We thank Catriona Byrne for her encouragement and invaluable editorial supervision, as well as Susanne Denskus for her invaluable technical assistance.

London and Sydney
September 2004

Marek Musiela
Marek Rutkowski

Note on the Second Printing

The second printing of the second edition of this book expands and clarifies further its contents exposition. Several proofs previously left to the reader are now included. The presentation of LIBOR and swap market models is expanded to include the joint dynamics of the underlying processes under the relevant probability measures. The appendix is completed with several frequently used theoretical results making the book even more self-contained. The bibliographical references are brought up to date as far as possible.

This printing corrects also numerous typographical errors and mistakes. We would like to express our gratitude to Alan Bain and Imanuel Costigan who uncovered many of them.

London and Sydney
August 2006

Marek Musiela
Marek Rutkowski

Preface to the First Edition

The origin of this book can be traced to courses on financial mathematics taught by us at the University of New South Wales in Sydney, Warsaw University of Technology (Politechnika Warszawska) and Institut National Polytechnique de Grenoble. Our initial aim was to write a short text around the material used in two one-semester graduate courses attended by students with diverse disciplinary backgrounds (mathematics, physics, computer science, engineering, economics and commerce). The anticipated diversity of potential readers explains the somewhat unusual way in which the book is written. It starts at a very elementary mathematical level and does not assume any prior knowledge of financial markets. Later, it develops into a text which requires some familiarity with concepts of stochastic calculus (the basic relevant notions and results are collected in the appendix). Over time, what was meant to be a short text acquired a life of its own and started to grow. The final version can be used as a textbook for three one-semester courses – one at undergraduate level, the other two as graduate courses.

The first part of the book deals with the more classical concepts and results of arbitrage pricing theory, developed over the last thirty years and currently widely applied in financial markets. The second part, devoted to interest rate modelling is more subjective and thus less standard. A concise survey of short-term interest rate models is presented. However, the special emphasis is put on recently developed models built upon market interest rates.

We are grateful to the Australian Research Council for providing partial financial support throughout the development of this book. We would like to thank Alan Brace, Ben Goldys, Dieter Sondermann, Erik Schlögl, Lutz Schlögl, Alexander Mürmann, and Alexander Zilberman, who offered useful comments on the first draft, and Barry Gordon, who helped with editing.

Our hope is that this book will help to bring the mathematical and financial communities closer together, by introducing mathematicians to some important problems arising in the theory and practice of financial markets, and by providing finance professionals with a set of useful mathematical tools in a comprehensive and self-contained manner.

Sydney
March 1997

Marek Musiela
Marek Rutkowski

Contents

Preface to the Second Edition	v
Note on the Second Printing	vii
Preface to the First Edition	ix

Part I Spot and Futures Markets

1 An Introduction to Financial Derivatives	3
1.1 Options	3
1.2 Futures Contracts and Options	5
1.3 Forward Contracts	6
1.4 Call and Put Spot Options	8
1.4.1 One-period Spot Market	9
1.4.2 Replicating Portfolios	10
1.4.3 Martingale Measure for a Spot Market	12
1.4.4 Absence of Arbitrage	13
1.4.5 Optimality of Replication	15
1.4.6 Change of a Numeraire	17
1.4.7 Put Option	18
1.5 Forward Contracts	19
1.5.1 Forward Price	20
1.6 Futures Call and Put Options	21
1.6.1 Futures Contracts and Futures Prices	21
1.6.2 One-period Futures Market	22
1.6.3 Martingale Measure for a Futures Market	23
1.6.4 Absence of Arbitrage	24
1.6.5 One-period Spot/Futures Market	26
1.7 Options of American Style	26
1.8 Universal No-arbitrage Inequalities	31

2	Discrete-time Security Markets	35
2.1	The Cox-Ross-Rubinstein Model	36
2.1.1	Binomial Lattice for the Stock Price	36
2.1.2	Recursive Pricing Procedure	38
2.1.3	CRR Option Pricing Formula	43
2.2	Martingale Properties of the CRR Model	46
2.2.1	Martingale Measures	47
2.2.2	Risk-neutral Valuation Formula	50
2.2.3	Change of a Numeraire	51
2.3	The Black-Scholes Option Pricing Formula	53
2.4	Valuation of American Options	58
2.4.1	American Call Options	58
2.4.2	American Put Options	60
2.4.3	American Claims	61
2.5	Options on a Dividend-paying Stock	63
2.6	Security Markets in Discrete Time	65
2.6.1	Finite Spot Markets	66
2.6.2	Self-financing Trading Strategies	66
2.6.3	Replication and Arbitrage Opportunities	68
2.6.4	Arbitrage Price	69
2.6.5	Risk-neutral Valuation Formula	70
2.6.6	Existence of a Martingale Measure	73
2.6.7	Completeness of a Finite Market	75
2.6.8	Separating Hyperplane Theorem	77
2.6.9	Change of a Numeraire	78
2.6.10	Discrete-time Models with Infinite State Space	79
2.7	Finite Futures Markets	80
2.7.1	Self-financing Futures Strategies	81
2.7.2	Martingale Measures for a Futures Market	82
2.7.3	Risk-neutral Valuation Formula	84
2.7.4	Futures Prices Versus Forward Prices	85
2.8	American Contingent Claims	87
2.8.1	Optimal Stopping Problems	90
2.8.2	Valuation and Hedging of American Claims	97
2.8.3	American Call and Put	101
2.9	Game Contingent Claims	101
2.9.1	Dynkin Games	102
2.9.2	Valuation and Hedging of Game Contingent Claims	108
3	Benchmark Models in Continuous Time	113
3.1	The Black-Scholes Model	114
3.1.1	Risk-free Bond	114
3.1.2	Stock Price	114
3.1.3	Self-financing Trading Strategies	118
3.1.4	Martingale Measure for the Black-Scholes Model	120

3.1.5	Black-Scholes Option Pricing Formula	125
3.1.6	Case of Time-dependent Coefficients	131
3.1.7	Merton's Model	132
3.1.8	Put-Call Parity for Spot Options	134
3.1.9	Black-Scholes PDE	134
3.1.10	A Riskless Portfolio Method	137
3.1.11	Black-Scholes Sensitivities	140
3.1.12	Market Imperfections	144
3.1.13	Numerical Methods	145
3.2	A Dividend-paying Stock	147
3.2.1	Case of a Constant Dividend Yield	148
3.2.2	Case of Known Dividends	151
3.3	Bachelier Model	154
3.3.1	Bachelier Option Pricing Formula	155
3.3.2	Bachelier's PDE	157
3.3.3	Bachelier Sensitivities	158
3.4	Black Model	159
3.4.1	Self-financing Futures Strategies	160
3.4.2	Martingale Measure for the Futures Market	160
3.4.3	Black's Futures Option Formula	161
3.4.4	Options on Forward Contracts	165
3.4.5	Forward and Futures Prices	167
3.5	Robustness of the Black-Scholes Approach	168
3.5.1	Uncertain Volatility	168
3.5.2	European Call and Put Options	169
3.5.3	Convex Path-independent European Claims	172
3.5.4	General Path-independent European Claims	177
4	Foreign Market Derivatives	181
4.1	Cross-currency Market Model	181
4.1.1	Domestic Martingale Measure	182
4.1.2	Foreign Martingale Measure	184
4.1.3	Foreign Stock Price Dynamics	185
4.2	Currency Forward Contracts and Options	186
4.2.1	Forward Exchange Rate	186
4.2.2	Currency Option Valuation Formula	187
4.3	Foreign Equity Forward Contracts	191
4.3.1	Forward Price of a Foreign Stock	191
4.3.2	Quanto Forward Contracts	192
4.4	Foreign Market Futures Contracts	194
4.5	Foreign Equity Options	197
4.5.1	Options Struck in a Foreign Currency	198
4.5.2	Options Struck in Domestic Currency	199
4.5.3	Quanto Options	200
4.5.4	Equity-linked Foreign Exchange Options	202

5	American Options	205
5.1	Valuation of American Claims	206
5.2	American Call and Put Options	213
5.3	Early Exercise Representation of an American Put	216
5.4	Analytical Approach	219
5.5	Approximations of the American Put Price	222
5.6	Option on a Dividend-paying Stock	224
5.7	Game Contingent Claims	226
6	Exotic Options	229
6.1	Packages	230
6.2	Forward-start Options	231
6.3	Chooser Options	232
6.4	Compound Options	233
6.5	Digital Options	234
6.6	Barrier Options	235
6.7	Lookback Options	238
6.8	Asian Options	242
6.9	Basket Options	245
6.10	Quantile Options	249
6.11	Other Exotic Options	251
7	Volatility Risk	253
7.1	Implied Volatilities of Traded Options	254
7.1.1	Historical Volatility	255
7.1.2	Implied Volatility	255
7.1.3	Implied Volatility Versus Historical Volatility	256
7.1.4	Approximate Formulas	257
7.1.5	Implied Volatility Surface	259
7.1.6	Asymptotic Behavior of the Implied Volatility	261
7.1.7	Marked-to-Market Models	264
7.1.8	Vega Hedging	265
7.1.9	Correlated Brownian Motions	267
7.1.10	Forward-start Options	269
7.2	Extensions of the Black-Scholes Model	273
7.2.1	CEV Model	273
7.2.2	Shifted Lognormal Models	277
7.3	Local Volatility Models	278
7.3.1	Implied Risk-Neutral Probability Law	278
7.3.2	Local Volatility	281
7.3.3	Mixture Models	287
7.3.4	Advantages and Drawbacks of LV Models	290
7.4	Stochastic Volatility Models	291
7.4.1	PDE Approach	292
7.4.2	Examples of SV Models	293

7.4.3	Hull and White Model	294
7.4.4	Heston's Model	299
7.4.5	SABR Model	301
7.5	Dynamical Models of Volatility Surfaces	302
7.5.1	Dynamics of the Local Volatility Surface	303
7.5.2	Dynamics of the Implied Volatility Surface	303
7.6	Alternative Approaches	307
7.6.1	Modelling of Asset Returns	308
7.6.2	Modelling of Volatility and Realized Variance	313
8	Continuous-time Security Markets	315
8.1	Standard Market Models	316
8.1.1	Standard Spot Market	316
8.1.2	Futures Market	325
8.1.3	Choice of a Numeraire	327
8.1.4	Existence of a Martingale Measure	330
8.1.5	Fundamental Theorem of Asset Pricing	332
8.2	Multidimensional Black-Scholes Model	333
8.2.1	Market Completeness	335
8.2.2	Variance-minimizing Hedging	337
8.2.3	Risk-minimizing Hedging	338
8.2.4	Market Imperfections	345

Part II Fixed-income Markets

9	Interest Rates and Related Contracts	351
9.1	Zero-coupon Bonds	351
9.1.1	Term Structure of Interest Rates	352
9.1.2	Forward Interest Rates	353
9.1.3	Short-term Interest Rate	354
9.2	Coupon-bearing Bonds	354
9.2.1	Yield-to-Maturity	355
9.2.2	Market Conventions	357
9.3	Interest Rate Futures	358
9.3.1	Treasury Bond Futures	358
9.3.2	Bond Options	359
9.3.3	Treasury Bill Futures	360
9.3.4	Eurodollar Futures	362
9.4	Interest Rate Swaps	363
9.4.1	Forward Rate Agreements	364
9.5	Stochastic Models of Bond Prices	366
9.5.1	Arbitrage-free Family of Bond Prices	366
9.5.2	Expectations Hypotheses	367
9.5.3	Case of Itô Processes	368

9.5.4	Market Price for Interest Rate Risk	371
9.6	Forward Measure Approach	372
9.6.1	Forward Price	373
9.6.2	Forward Martingale Measure	375
9.6.3	Forward Processes	378
9.6.4	Choice of a Numeraire	379
10	Short-Term Rate Models	383
10.1	Single-factor Models	384
10.1.1	Time-homogeneous Models	384
10.1.2	Time-inhomogeneous Models	394
10.1.3	Model Choice	399
10.1.4	American Bond Options	401
10.1.5	Options on Coupon-bearing Bonds	402
10.2	Multi-factor Models	402
10.2.1	State Variables	403
10.2.2	Affine Models	404
10.2.3	Yield Models	404
10.3	Extended CIR Model	406
10.3.1	Squared Bessel Process	407
10.3.2	Model Construction	407
10.3.3	Change of a Probability Measure	408
10.3.4	Zero-coupon Bond	409
10.3.5	Case of Constant Coefficients	410
10.3.6	Case of Piecewise Constant Coefficients	411
10.3.7	Dynamics of Zero-coupon Bond	412
10.3.8	Transition Densities	414
10.3.9	Bond Option	415
11	Models of Instantaneous Forward Rates	417
11.1	Heath-Jarrow-Morton Methodology	418
11.1.1	Ho and Lee Model	419
11.1.2	Heath-Jarrow-Morton Model	419
11.1.3	Absence of Arbitrage	421
11.1.4	Short-term Interest Rate	427
11.2	Gaussian HJM Model	428
11.2.1	Markovian Case	430
11.3	European Spot Options	434
11.3.1	Bond Options	435
11.3.2	Stock Options	438
11.3.3	Option on a Coupon-bearing Bond	441
11.3.4	Pricing of General Contingent Claims	444
11.3.5	Replication of Options	446
11.4	Volatilities and Correlations	449
11.4.1	Volatilities	449

11.4.2	Correlations	451
11.5	Futures Price	452
11.5.1	Futures Options	453
11.6	PDE Approach to Interest Rate Derivatives	457
11.6.1	PDEs for Spot Derivatives	457
11.6.2	PDEs for Futures Derivatives	461
11.7	Recent Developments	465
12	Market LIBOR Models	469
12.1	Forward and Futures LIBORs	471
12.1.1	One-period Swap Settled in Arrears	471
12.1.2	One-period Swap Settled in Advance	473
12.1.3	Eurodollar Futures	474
12.1.4	LIBOR in the Gaussian HJM Model	475
12.2	Interest Rate Caps and Floors	477
12.3	Valuation in the Gaussian HJM Model	479
12.3.1	Plain-vanilla Caps and Floors	479
12.3.2	Exotic Caps	481
12.3.3	Captions	483
12.4	LIBOR Market Models	484
12.4.1	Black's Formula for Caps	484
12.4.2	Miltersen, Sandmann and Sondermann Approach	486
12.4.3	Brace, Gařarek and Musiela Approach	486
12.4.4	Musiela and Rutkowski Approach	489
12.4.5	SDEs for LIBORs under the Forward Measure	492
12.4.6	Jamshidian's Approach	495
12.4.7	Alternative Derivation of Jamshidian's SDE	498
12.5	Properties of the Lognormal LIBOR Model	500
12.5.1	Transition Density of the LIBOR	501
12.5.2	Transition Density of the Forward Bond Price	503
12.6	Valuation in the Lognormal LIBOR Model	506
12.6.1	Pricing of Caps and Floors	506
12.6.2	Hedging of Caps and Floors	508
12.6.3	Valuation of European Claims	510
12.6.4	Bond Options	513
12.7	Extensions of the LLM Model	515
13	Alternative Market Models	517
13.1	Swaps and Swaptions	518
13.1.1	Forward Swap Rates	518
13.1.2	Swaptions	522
13.1.3	Exotic Swap Derivatives	524
13.2	Valuation in the Gaussian HJM Model	527
13.2.1	Swaptions	527
13.2.2	CMS Spread Options	527

13.2.3	Yield Curve Swaps	529
13.3	Co-terminal Forward Swap Rates	530
13.3.1	Jamshidian's Model	535
13.3.2	Valuation of Co-terminal Swaptions	538
13.3.3	Hedging of Swaptions	539
13.3.4	Bermudan Swaptions	540
13.4	Co-initial Forward Swap Rates	541
13.4.1	Valuation of Co-initial Swaptions	544
13.4.2	Valuation of Exotic Options	545
13.5	Co-sliding Forward Swap Rates	546
13.5.1	Modelling of Co-sliding Swap Rates	547
13.5.2	Valuation of Co-sliding Swaptions	551
13.6	Swap Rate Model Versus LIBOR Model	552
13.6.1	Swaptions in the LLM Model	553
13.6.2	Caplets in the Co-terminal Swap Market Model	557
13.7	Markov-functional Models	558
13.7.1	Terminal Swap Rate Model	559
13.7.2	Calibration of Markov-functional Models	562
13.8	Flesaker and Hughston Approach	565
13.8.1	Rational Lognormal Model	568
13.8.2	Valuation of Caps and Swaptions	569
14	Cross-currency Derivatives	573
14.1	Arbitrage-free Cross-currency Markets	574
14.1.1	Forward Price of a Foreign Asset	576
14.1.2	Valuation of Foreign Contingent Claims	580
14.1.3	Cross-currency Rates	581
14.2	Gaussian Model	581
14.2.1	Currency Options	582
14.2.2	Foreign Equity Options	583
14.2.3	Cross-currency Swaps	588
14.2.4	Cross-currency Swaptions	599
14.2.5	Basket Caps	602
14.3	Model of Forward LIBOR Rates	603
14.3.1	Quanto Cap	604
14.3.2	Cross-currency Swap	606
14.4	Concluding Remarks	607
<hr/>		
Part III APPENDIX		
<hr/>		
A	An Overview of Itô Stochastic Calculus	611
A.1	Conditional Expectation	611
A.2	Filtrations and Adapted Processes	615
A.3	Martingales	616

A.4	Standard Brownian Motion	617
A.5	Stopping Times and Martingales	621
A.6	Itô Stochastic Integral	622
A.7	Continuous Local Martingales	625
A.8	Continuous Semimartingales	628
A.9	Itô's Lemma	630
A.10	Lévy's Characterization Theorem	633
A.11	Martingale Representation Property	634
A.12	Stochastic Differential Equations	636
A.13	Stochastic Exponential	639
A.14	Radon-Nikodým Density	640
A.15	Girsanov's Theorem	641
A.16	Martingale Measures	645
A.17	Feynman-Kac Formula	646
A.18	First Passage Times	649
References		657
Index		707