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Jean-Luc Prigent

Weak Convergence of Financial Markets

With 8 Figures
and 1 Table

Springer

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To My Family and S.

Preface

Motivation

- One of the main problem treated in this book is the following :

  Continuous and discrete time financial models are at best approximations of the reality. So, it seems important to connect them and to compare their predictions when, in the discrete time setting, periods between trades shrink to zero. But does the convergence of stocks prices imply the convergence of optimal portfolio strategies, derivatives prices and hedging strategies ?

  We can alternatively ask the following question :

  Consider two investors who estimate stock prices from statistical data, one in a discrete time setting, the other one in continuous time. Suppose they agree on the stock price distribution, for example a GARCH model in discrete time for the first one and a Hull and White type model for the second one (it means that the discrete time periods are sufficiently small to accept that the distributions of the stock log returns are equal). When for instance they determine the no-arbitrage prices, each one for his own model, do they necessarily agree also for example on options prices or spreads ?

  As it will be seen in this book, the answer is not straightforward. For example, convergence of options prices is typically proved for binomial tree (for example, the well known Cox-Ross-Rubinstein derivation of the Black-Scholes formula) or suitable multinomial trees and is established in the complete case. However, real markets are usually incomplete. This may induce "instability" of financial variables or instruments due to convergence problems within various financial models. This point is illustrated in chapter 2 for optimal portfolio policies, option pricing and hedging strategies.

  This lack of robustness for some basic approximations shows that we must be particularly cautious when dealing with convergence problems.

- It is often easier to derive analytic or numerical results in discrete time than in continuous time (or vice versa). Hence a second purpose is to recall some basic approximations which are of particular interest to build numerical algorithms. They can be applied for the pricing of American, Asian and barrier options on stocks or indexes and to approximate bonds and interest rates derivatives.

- Finally, we have to make a choice : *what type of convergence should we use ?*

  As it is well-known, convergence in distribution, also called "*weak convergence*" is a convenient tool in many statistical studies. It further allows to analyze stochastic phenomena without specifying a particular probability space: often in practice, only the set of values of the observed stochastic processes is involved. "Weak convergence" refers here to the convergence in distribution for stochastic processes treated as random elements of function spaces. Despite its greater complexity (due to "*tightness condition*") when compared to the weak convergence for finite-dimensional distributions, the "*functional*" weak convergence is useful: contrary to the former mode, it can guarantee convergence of exotic option prices, such as Asian options which involve the whole path of the stock process.

  To summarize, the purpose of this book is to apply the theory of weak convergence of stochastic processes to the study of financial markets.

Readership

This book assumes the reader has a good knowledge of probability theory in continuous time. It is aimed at an audience with a sound mathematical background. It supposes also that basic financial theory, such as valuation and hedging of derivatives, is already known. *However* :

- In the first chapter, basic notions and definitions of stochastic processes are first recalled. Second, an overview of the theory of weak convergence of semimartingales is provided. In particular, a guideline is given for the weak convergence of stochastic integrals and contiguity properties.

- Along the second chapter, the standard notions and properties of the financial markets theory are recalled (but not detailed).

- Finally, the emphasis throughout the third chapter is on presenting the basic discrete models and their continuous time limits. The focus remains on a survey about multinomial approximations and more generally about computing problems with lattices for different types of options. Other approximations such as ARCH models...are also introduced and detailed. Nevertheless, a perfect knowledge of the first two chapters is not fully required.

Book Structure

- The first chapter tries to answer the question :

How to prove that a sequence $(X_n)_n$ of stochastic processes weakly converges to a given stochastic process X ?

This mathematical chapter is only a guide to the reader. While main results are included, the proofs are not provided, as they are already excellent treatment of this theory readily available :

- The books of Dellacherie and Meyer [108] present the general properties of stochastic processes (volume I) and martingales (volume II), Ethier and Kurtz [148] deals with Markov processes, Elliott [144], Kopp [250] and Rogers and Williams [365] introduce the stochastic integration. The books of McKean [286], Chung and Williams [75] and Karatzas and Shreve [236] dal with Brownian motion and continuous martingales. The book of Protter [351] gives a very clear presentation of semimartingales, stochastic integration and stochastic differential equations.

- Concerning main results of weak convergence of semimartingales, it is referred to Jacod and Shiryaev's book [214].

- Nevertheless, with respect to this latter book, two parts are added :

1) A special emphasis on the weak convergence of sequences of triangular arrays, which are of particular interest, when dealing with convergence problems from discrete time to continuous time models.

2) A survey of main results concerning weak convergence of sequences of stochastic integrals and solutions of stochastic differential equations (see also the new version of Jacod and Shiryaev's book to appear in 2003).

- The second chapter deals with the following question :

What are the main problems that we encounter when examining weak convergence of financial markets ?

Thus this chapter introduced the results about weak convergence of :

- Optimal portfolio policies for utility maximizing investors.

- Option prices, in particular convergence problems of bid-ask spreads.

- Hedging strategies which duplicate options in complete financial markets or hedging strategies which minimize the locally quadratic risk when facing incomplete markets.

A survey of basic results of financial theory is included but not detailed since many books are also available :

- For the main notions, among others, Duffie [124][125][126], Bingham and Kiesel [37], Kwok [261], Elliott and Kopp [145], Lamberton and Lapeyre [264], Nielsen [321], Björk [39].

- More particularly, Pliska [339] introduces all of the main financial concepts for the discrete time case. Musiela and Rutkowski [312] deal in particular with the theory of bond markets and term structure models. In Jeanblanc-Picque and Dana [220], the equilibrium approach is detailed. Shiryaev [381] introduces a large variety of stochastic models.

Hence, in this chapter, we focus on convergence results. Some particular proofs are fully detailed to show how weak convergence results of the first chapter can be applied.

- The third chapter reviews a list of results to solve the following problem :

How to construct in practice a sequence $(X_n)_n$ of stochastic processes which weakly converges to a given stochastic process X ?

Although it is not a purely "numerical" chapter, many of standard approximations of basic continuous time processes are recalled :

- Its first part contains some general results about approximations of solutions of stochastic differential equations (standard and backward).

- Its second part is devoted to standard lattice models, when approximating diffusions. Binomial and trinomial schemes are especially examined when the continuous time limit process is driven by a Brownian motion. For most of these models, the discrete time subdivision of the time interval is deterministic.

- A third part proposes other models for example diffusions with jumps. This class of processes contains Lévy processes and in particular subordinators which are of particular interest when examining dynamics of high frequency data. Both deterministic and random discretizations are studied. By considering sequences of random times, the latter ones allow in particular to examine problems of portfolio rebalancing.

- Finally, a list of some standard approximations of interest rate models is provided: factor models as well as Heath-Jarrow-Morton type models and Market models are briefly reviewed.

Final Word and Acknowledgments

This book is an attempt to summarize the main convergence results about financial markets which are known at present. It is focussed on robustness of financial instruments under convergence of discrete time to continuous time financial markets. In particular, it indicates option pricing rules that are stable under convergence of the underlying assets. This feature reduces the model risk when we must choose between discrete time or continuous time to describe asset prices dynamics.

While some of the quoted results concern more academics than practitioners, it seems important to underline main features of convergence, first of all that approximating models do converge. Both "pure mathematical" speed (in the spirit of the famous *Central Limit Theorem*) and computational speed must be analyzed. Obviously, all convergence problems are not yet solved. Further extensions are still in progress both in the mathematical field (to take more dependency properties into account, to obtain functional speed of convergence...) and also in the financial theory (search of other algorithms to simulate financial variables, study of more general option pricing or portfolio problems taking account of market imperfections such as trading strategies which are unavailable in continuous time...).

I hope that this book will contribute to stimulate new research on the sometimes awkward (nevertheless fascinating ?) weak convergence world and financial theory.

While I cannot thank all the people for supports and useful discussions since I began to study the financial theory, I want to mention in particular my colleagues of the research department THEMA, the members of HSBC-CCF and in particular Eric Baesen and Jean-François Boulier, also Nicole El Karoui who encouraged me some years ago to work in financial mathematics, especially to examine weak convergence problems, Patrick Navatte, Patrice Poncet and Yves Simon who inclined me to discover and study more applied features of the financial market theory and Monique Jeanblanc whose legendary kindness is such that I have never hesitated before telling her with my problems.

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