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Models of the Real Projective Plane

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Computer Graphics of Steiner and Boy Surfaces

With 46 Figures and 64 Color Plates

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Preface

by *Egbert Brieskorn*

If you feel attracted by the beautiful figure on the cover of this book, your feeling is something you share with all geometers. "It is the delight in figures in a higher sense which distinguishes the geometer". This is a well known statement of the algebraic geometer Alfred Clebsch. Throughout the history of our science, great men like Plato and Kepler and Poincaré were inspired by the beauty and harmony of such figures. Whether we see them as elements or representations of some universal harmony or simply look at them as a structure accessible to our mind by its very harmony, there is no doubt that the contemplation and the creation of beautiful geometric figures has been and will be an integral part of creative mathematical thought.

Perception of beauty is not the work of the intellect alone – it needs sensitivity. In fact, it does need sensual experience. To seperate the ideas from the appearances, the structure from the surface, analytical thinking from holistic perception and science from art does not correspond to the reality of our mind and is simply wrong if it is meant as an absolute distinction. It would deprive our creative thought of one of its deepest sources of inspiration.

I remember a visit of the blind mathematician B. Morin in Göttingen. He gave us a lecture on the eversion of the sphere, the process which he had discovered for turning the sphere inside out by a continuous family of immersions. At that time, he had no analytic description of this very complicated process, but he had a very precise qualitative picture of its geometry in his mind, and he had pictures of models made according to his instruction. I was deeply impressed by the beauty of these figures, and I was moved by the fact that he had been able to see all this beautiful and complex geometry which we could visualize only with great difficulty even after it was shown to us.

In Göttingen we had a very nice collection of mathematical models dating from the days of Felix Klein and David Hilbert. You can find photographs of some of them in a very nice book of Gerd Fischer entitled "Mathematical models", which was published by Vieweg. Among these models there is one made in 1903 showing the surface of Boy. This is a surface obtained by an immersion of the real projective plane in 3-dimensional space. At the time when it was made it was known that the projective plane cannot be embedded as a smooth surface in 3-space. This is so because any smooth closed surface in 3-space divides the space into an interior part and an exterior. This implies that it is orientable: If you stand in the exterior with your feet on the surface, you know what it means to turn left. But the real projective plane is not orientable. This was a very remarkable discovery of Felix Klein in 1874, and it implies that the projective plane $\mathbb{P}^2$ cannot be embedded in the 3-dimensional space $\mathbb{R}^3$. So if we want to visualize $\mathbb{P}^2$ by means of a surface in $\mathbb{R}^3$, we have to be contented with less than a smooth surface. For instance we should admit that different parts of the surface penetrate each other. If the parts themselves are smooth and penetrate each other transversally, we call the surface transversally immersed. It is by no means obvious that there is a presentation of $\mathbb{P}^2$ as an immersed surface in $\mathbb{R}^3$. If there was none, we would be forced to allow the surface to have some singular points, i.e. points where it is not smooth. Such representations of $\mathbb{P}^2$ by surfaces with selfpenetrations and singular points were known to exist. One of them is a beautiful surface discovered by the German geometer Jacob Steiner. He called it his "Roman surface", because he discovered some of its properties during a stay at Tome in 1844. You can see a picture of this surface on **Plate 57** of this book and you can buy a pretty 3-dimensional model of it from Vieweg. It has tetrahedral symmetry, with six lines of selfpenetration issuing from the center of the tetrahedron in the direction of the mid-points. These singular points are of the simplest possible type. They are so called "Whitney-umbrellas" (**Plate 23**). These singularities are the simplest singularities (apart from selfpenetration) which can occur in this situation, and they are stable: a Whitney umbrella will not go away if the surface is deformed slightly.

Maybe it was the stability of these singularities that made David Hilbert suspicious so that he thought it would be impossible to present the projective plane $\mathbb{P}^2$ by an immersed surface without singular points. But the famous mathematician David Hilbert was mistaken. In 1901 his student W. Boy produced two surfaces which are immersions of $\mathbb{P}^2$ in $\mathbb{R}^3$. One of them is particularly pretty. It has a 3-fold axis of symmetry and this is the Boy surface which is the main theme of this book.

Of course, B. Morin knew Boy's surface when he visited us in Göttingen. He must have had a very clear picture of it in his mind. Nevertheless he asked us to take our model out of the show-case so that he could touch it and feel whether its curvature was beautiful. It is this kind of sensitivity for the genesis and transformation of forms which I think is necessary for the creation of ideas such as those of B. Morin or R. Thom leading to the beautiful constructions of F. Apéry that you find in this book.

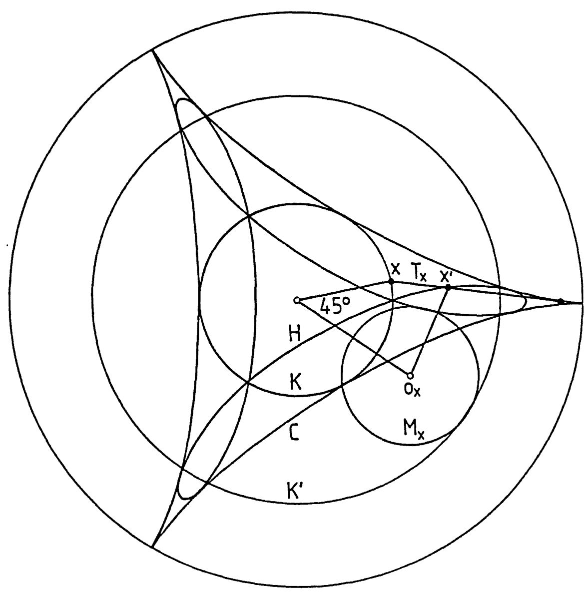
Boy's surface, like Morin's halfway model of the eversion of the sphere at the time of his visit in Göttingen, was only described in terms of a qualitative geometry. Such descriptions are quite satisfactory from the point of view of differential topology. In fact since the days of H. Poincaré we know that in many situations such a qualitative description is all we can hope for. However, there is an older tradition in mathematics which asks for a more definite way of describing solutions of a problem. In algebraic or analytic geometry, a surface imbedded in 3-space is to be described by an equation or a parametrization by nice analytic functions such as polynomials or by some beautiful geometric construction. In a sense, a geometric figure can only be considered as a unique entity worthy of this name "figure" in the ancient tradition if it is well-defined in this way. This is what Kepler meant when he said: "That which is construction in geometry is consonance in music". Thus our desire for harmony in a geometric figure will only be fulfilled if we see it described in such a way. This beautiful book of F. Apéry tells the exciting history of the discovery of such a description for the surface of Boy.

It begins with very qualitative topological and combinatorical descriptions of the projective plane and the surface of Boy, and it shows us nice descriptions in terms of equations and parametrizations obtained by B. Morin 1978 (**Plate 33**), J.-P. Petit and J. Souriau (1981) (**Plates 31, 32**), J. F. Hughes 1985 (**Plate 34**) and J. Bryant (**Plate 35**). He then presents a very nice solution of his own originating from an idea of B. Morin. The idea is to construct the apparent contour of the Boy surface seen from infinity in the direction of its 3-fold axis by interpreting parts of it as a deformation of the "folded handkerchief singularity" mapping (x, y) to (x^2, y^2). This leads to a parametrization of a Boy surface in 3-space by 3 polynomials of degree 4 (proposition 4 on page 60, **Plates 36–38**). However, all these solutions have slight deficiencies. For instance the equation describing the surface may have other real zeros different from the points of the Boy surface.

But finally the author presents a truly perfect solution. It describes the Boy surface by a beautiful geometric construction, which is subtle and elementary, in the best tradition of classical geometry. In order to make it easy for you to understand his analysis, let me explain the construction in purely geometric terms. It generates the surface by a 1-parameter family of ellipses obtained as follows.

Let L be a line and p a point on L. This line will become the 3-fold axis of symmetry, and the pole p will be a point where L intersects the surface. Let P be the plane orthogonal to L passing through p, and let Q be a plane parallel to P at distance 1. In Q draw a circle K of radius $\sqrt{2}/3$ with axis L. Then draw a threecuspidal hypocycloid C such that K is its inscribed circle. C may be obtained as trace of a point of the circumference of a circle of radius $\sqrt{2}/3$ rolling inside a circle of radius $\sqrt{2}$ concentric with K. The curve C will be the apparent contour of the Boy surface that we are going to construct, if we project the surface from the pole p to the plane Q.

Next we draw an elongated hypocycloid H in the plane Q as follows. Let K' be a circle of radius 1 in Q concentric with K. Let M be a moving circle of radius 1/3 which rolls without gliding inside K'. Let t_0 be a point in the moving plane of M at distance $\sqrt{2}/3$ from the centre of M. As M rolls, t_0 will trace out an elongated hypocycloid in the plane Q. This curve will have three ordinary double points, and they lie on K. We choose the initial position of t_0 so that these double points are situated on the tangents of the 3 cusps of the 3-cuspidal hypocycloid C, as indicated by the following figure.



Now we define a map $K \to H$. Let x be a point moving on K in the positive sense. Let o_x be the centre of the moving circle M_x, where o_x is moving on a circle of radius 2/3 with the same angular velocity as x, but trailing behind $45°$. The point t_0 for M_x determines a unique point x' on the elongated hypocycloid H, and our map is defined by associating this point x' to x.

It follows from the construction that x and x' lie on a tangent T_x to the hypocycloid C. Let P_x be the plane passing through this tangent T_x and the pole p. We define an ellipse E_x in the plane P_x by the following three conditions:

(a) E_x passes through x and x'.
(b) E_x is tangent to P in p.
(c) The distance d of x and x' and the maximal distance δ of a point on E_x from P are related as follows:

$$\delta = (1 - 9d^2/16)^{-1}.$$

These conditions define E_x uniquely.

And now our construction comes to its end: The surface S to be constructed is simply the union of all ellipses E_x. As x moves on K, the ellipses sweep out the surface. This is the Boy surface constructed by F. Apéry. Apéry proves that it is exactly the set of real zeroes of a polynomial equation of degree 6 which he can write down explicitly. Moreover he gives a nice description of the transversal immersion $\mathbb{P}^2 \to S \subset \mathbb{R}^3$. The ellipses are the images of a pencil of projective lines passing through the point in $\mathbb{P}^2$ which corresponds to the pole p. If we interpret $\mathbb{P}^2$ as quotient of the 2-sphere S^2 identifying antipodal points, the corresponding map $S^2 \to \mathbb{R}^3$ has a nice description in terms of trigonometric functions, and the inverse images of the ellipses are the great circles passing through a pair of antipodal points. **Plates 39–41** show this pretty surface.

Apéry's construction is beautiful, elementary, but subtle and extremely elegant. Moreover there is a similar construction for the Roman surface of Steiner, and one can pass from one surface to the other by a continuous family of surfaces, such that at a certain moment the six Whitney-umbrella-type singularities on the deformations of the Roman surface annihilate each other in pairs by a nice process called hyperbolic confluence. This process is illustrated on **Plates 28–30**, and the family of surfaces is shown on **Plates 51–56.**

Finally Apéry generalizes his approach and gets similar descriptions of other surfaces. Among them is the beautiful halfway model of B. Morin which you see on the cover of this book. All this is sheer beauty, and I hope that you will be delighted.

Introduction

In this monograph we study the real projective plane from the topological point of view, more precisely we look at its representations in $\mathbb{R}^3$. Provided that we give a sufficiently rich crop of examples, such an excursion into descriptive zoology can serve as the pretext for a didactic introduction to theories with applications extending far beyond the field of the initial investigation.

Guided by this principle we have tried throughout to strike a balance between abstract definitions (from final year undergraduate or first year postgraduate mathematics) and related examples which are enlivened by drawings and computer graphics. The color plates were created using the PS 300 computer of the Laboratoire de Cristallographie de l'Institut de Biologie Moléculaire et Cellulaire in Strasbourg; R. Ripp provided much valuable assistance in taming this remarkable beast.

As soon as it was realized that the real projective plane is nonorientable and can not be embedded in $\mathbb{R}^3$, the hunt was on for tractable representations in three dimensions. The various theoretical stages leading to W. Boy's construction in 1901 of the self-transversal immersion which now bears his name can be conceptualized by the following results of H. Whitney:

(i) any closed surface embeds in $\mathbb{R}^4$. We can obtain an embedding of the projective plane in $\mathbb{R}^4$ from the real Veronese mapping, whose image lies in a 4-dimensional sphere, by stereographically projecting the sphere onto $\mathbb{R}^4$ (sec. 2.3).

(ii) any closed surface embedded or immersed in $\mathbb{R}^4$ can be projected to a locally stable surface in $\mathbb{R}^3$, in other words one whose only singularities are Whitney umbrellas (sec. 2.3). Starting from the embedding described above of the real projective plane in $\mathbb{R}^4$, we find two surfaces originally discovered by Steiner: the Roman surface and the cross-cap (sec. 1.3). Another result of Whitney [WH1] shows that it is impossible to construct an immersion of the projective plane in $\mathbb{R}^3$ by projecting an embedding in $\mathbb{R}^4$.

(iii) Whitney umbrellas occur in pairs on the locally stable image of a closed surface in $\mathbb{R}^3$, and can be eliminated in pairs by means of two generic processes called elliptic and hyperbolic confluence of Whitney umbrellas (sec. 2.3). Starting with the cross-cap, a hyperbolic confluence of the two umbrellas leads to the first of the immersions constructed by Boy (**Plates 115 and 116** [FI]). If, on the other hand, we start from the Roman surface, we can then retain an axis of threefold symmetry throughout the deformation, and the elimination of the six umbrellas gives the Boy surface with threefold symmetry (**Plates 39–41**).

We remark that these two constructions in fact lead to the same geometrical object, as we shall show in section 3.2.

We begin chapter 1 by defining surfaces and giving some of their basic properties, the main aim of this chapter is to make an inventory of representations of the real projective plane in $\mathbb{R}^3$, paying particular attention to the Steiner surfaces.

In chapter 2 we explain the concepts of embedded and immersed surfaces in $\mathbb{R}^3$; our goal is to construct a smooth polynomial immersion of $\mathbb{P}^2$ in $\mathbb{R}^3$. We find an immersed surface which is not the real zero-set of any polynomial; there is an irreducible polynomial which vanishes on this surface, but it also vanishes elsewhere in $\mathbb{R}^3$.

As in (iii) above, we then construct a C^1-immersion of $\mathbb{P}^2$ in $\mathbb{R}^3$ by eliminating the six umbrellas from the Steiner Roman surface. The image of this immersion is the real zero-set of a sextic polynomial.

As an interlude, in section 2.1, we give explicit polynomials whose real zero-sets are the closed orientable surfaces in $\mathbb{R}^3$, and also a sextic polynomial whose real zero-set is the image of a smooth self-transversal immersion of the Klein bottle in $\mathbb{R}^3$; this last example is based on an idea of L. Siebenmann.

Chapter 3 is devoted to a classification of certain immersed projective planes in S^3 which are related to the Boy immersion; a final list consists of two examples.

In section 3.3 we obtain the halfway model of an eversion of the sphere described by B. Morin, as a subset of an algebraic surface of degree eight.

We have only introduced abstract mathematical ideas in so far as this was necessary for an understanding of the examples; and we have framed them in such a way that our explanations can be followed by a motivated honours student. We can not hope to give an exhaustive account of this subject and so we refer the interested reader to more technical and more advanced sources.

It was at a showing of a series of slides of surfaces on the occasion of the Symposium on singularities in Warsaw in 1985 that Professor E. Brieskorn offered to put me in touch with the publisher Vieweg in order to produce a book with color plates. In addition to having been at the origin of this book E. Brieskorn was also kind enough to read the manuscript and to allow me to profit from his pertinent comments.

It remains for us to thank A. Flegmann for having rendered this text comprehensible in English.

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