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Pseudo-Regularly Varying Functions and Generalized Renewal Processes

 Springer

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Preface

Renewal theory is a branch of probability theory rich in fascinating mathematical problems and also in various important applications. On the other hand, regular variation of functions is a property that plays a key role in many fields of mathematics. One of the main aims of this book is to exhibit some fruitful links between these two areas via a generalized approach to both of them.

The core of renewal theory is to study (so-called) renewal processes and their probabilistic and statistical characteristics. One of the most cited examples in renewal theory is the following, which deals with the life span of a light bulb.

Renewal Process Assume one has a light bulb in a room and one turns it on, keeping the light bulb working until it fails, after which it is replaced with a new light bulb. If ξ_i denotes the life span of the i th bulb, then $S_n = \xi_1 + \dots + \xi_n$ represents the total life span of the first n bulbs and

$$N_t = \max\{n : S_n \leq t\} \tag{1}$$

is the number of bulbs needed until time t . $\{N_t\}$ is called a renewal counting process constructed from the sequence $\{S_n\}$.

Of course, the bulbs in this example can be exchanged by any other expendable resource, which makes the example more realistic and more attractive from the point of view of applications. For example, consider arriving customers waiting in a queue until one of the servers is free to serve him/her. The arrival counting process to the queue is usually assumed to be a renewal counting process $\{N_t\}$, where the $\{\xi_i\}$ are the inter-arrival times between the customers.

Next we give some further examples of stochastic processes which are related to renewal processes and which occur both in pure and applied mathematics.

Reward Process Let some (random) event occur from time to time. Imagine that someone experiences a reward at each occurrence of the event. Let r_i be the reward earned at the time of the i th occurrence of the event. Denoting by N_t the number of occurrences of the event up to time t , we again see a renewal process with ξ_i being

the time between the $(i - 1)$ th and i th occurrence of the event. Then

$$Y_t = \sum_{i=1}^{N_t} r_i$$

is called a reward process, which is a special case of a (so-called) compound renewal process.

Treating the r_i 's as penalties rather than rewards we arrive at several other settings. One of the classical examples here arises in a risk model describing the evolution of the capital of an insurance company which experiences two opposing cash flows: incoming cash premiums and outgoing claims. Premiums from the customers may arrive at a constant rate $c > 0$ (say) and claims $\{r_i\}$ occur according to a counting process $\{N_t\}$, that is, N_t is the number of claims up to time t . So, for an insurer who starts with initial surplus x ,

$$X_t = x + ct - \sum_{i=1}^{N_t} r_i, \quad t \geq 0,$$

represents the capital at time t .

General Counting Process There are many situations similar to what has been described above. Let, for example, ξ_i be the time passed between the $(i - 1)$ th and i th record in a sport discipline. Then N_t , again, denotes the number of records up to time t and is, in general, called a counting process. Records can also have a negative meaning, for example losses caused by catastrophic events such as hurricanes.

Such a counting process can also be seen in pure mathematics. Let (say) S_n be the n th prime number. Then N_t (usually denoted by $\pi(t)$ in this case) plays a crucial role not only in number theory, but also in applications like cryptography when estimating the security of protocols used to transmit data. An example where a renewal process explicitly occurs in cryptomachines is discussed in Gut [159].

The central object of all the above models can be reduced to an investigation of the behavior of an underlying counting process $\{N_t\}$. In probability theory, this can effectively be done under the classical assumptions that the $\{\xi_i\}$ are

- (a) independent,
- (b) identically distributed,
- (c) positive

random variables. But what about the asymptotic behavior of N_t in cases where these assumptions fail? Such situations are not rare at all. Imagine, for example, that a catastrophic event occurs and leads to many claims for damages. Then the times $\{\xi_i\}$ between the claims become dependent, since many of the policy holders will report to an insurance company at (roughly) the same time. Another example, in which Assumption (b) fails, is related to the so-called alternating renewal process

$\{N_t\}$ constructed from a sequence

$$S_1 = \xi_1, \quad S_2 = \xi_1 + \eta_1, \quad S_3 = \xi_1 + \eta_1 + \xi_2, \quad S_4 = \xi_1 + \eta_1 + \xi_2 + \eta_2, \dots,$$

where the $\{\xi_i\}$ are independent random variables with distribution function F_ξ and the $\{\eta_i\}$ are independent random variables with distribution function F_η . This kind of renewal process occurs in chromatographic models or in describing the motion of water in a river (see, e.g., Gut [159] for a short discussion and further references). Finally, Assumption (c) may fail if, for example, $\{\xi_i\}$ describes a money flow including both expenses and incomes, so that ξ_i could attain both positive and negative values.

Formula (1) nevertheless defines a process $\{N_t\}$, called a *generalized renewal process* in this case, even if one of the Assumptions (a)–(c) fails. Moreover, one can introduce and study some other important functionals of $\{S_n\}$ here. For example, viewing $N_t + 1$ defined via (1) as the *first time* when $\{S_n\}$ exceeds the level t , one can also consider L_t , the *last passage time* across the level t . There are other natural functionals, e.g., T_t , the total time spent by $\{S_n\}$ below the level t , etc. The just mentioned three functionals coincide if Assumption (c) holds, but otherwise they are different. All these functionals as well as many others are also called generalized renewal processes. So, generalized renewal processes arise either if N_t is constructed by (1), but one of the properties (a)–(c) fails, or if other functionals of $\{S_n\}$ (similar to N_t) are considered (we shall specify later what kind of functionals we have in mind when studying generalized renewal processes).

One of the basic questions studied in all models including renewal processes concerns the asymptotic behavior of N_t as $t \rightarrow \infty$ which, in turn, is determined by the asymptotic behavior of S_n as $n \rightarrow \infty$ (we shall clarify what we mean by “asymptotic behavior” later in this book). Therefore one may expect that the asymptotic properties of the models for (generalized) renewal processes can be derived from the asymptotic behavior of $\{S_n\}$. In turn, it is also to be expected that one can argue vice versa, i.e., that the asymptotic behavior of $\{S_n\}$ is determined by that of $\{N_t\}$. In this case, one would be able to derive statistical properties of the ξ_i ’s by observing the counting process $\{N_t\}$. So, $\{S_n\}$ and $\{N_t\}$ may be viewed as “dual objects” in a certain sense.

Generally, we call two objects *dual* if their asymptotic properties are related to each other as indicated above, that is, if a limit result for the first object implies a corresponding one for the second object, and vice versa. The duality of $\{N_t\}$ and $\{S_n\}$ has been proved in Gut et al. [160] for the classical setting where all the Assumptions (a)–(c) hold. We would like to mention that Doob [111] predicted this duality property in 1948, but until very recently the limit theorems for renewal processes and those for sums of random variables developed independently.

Counting processes and their dual (or “renewal”) processes are often observed in number theory. For example, the total number $\pi(x)$ of prime numbers up to x and the n -th prime number p_n are dual objects, since $\pi(p_n) = n$. This duality is reflected by the *prime number theorem* stating that $\pi(x) \sim x / \ln(x)$ as $x \rightarrow \infty$ and

the asymptotic behavior $p_n \sim n \ln(n)$ as $n \rightarrow \infty$; note that $n \ln(n)$ is inverse to $x/\ln(x)$ in a certain sense.

Probability theory provides other examples of dual objects. One of them is the number μ_n of records until time n in a sequence of random variables and the magnitude τ_n of the n -th record. Obviously $\mu_{\tau_n} = n$ and this duality allows one to study the asymptotic behavior of μ_n and τ_n simultaneously. By the way, the second property of inverse functions fails in both cases since, in general, $\tau_{\mu_n} \neq n$ and $\tau_{\pi(n)} \neq n$.

Another example is the duality between the tail $\bar{F} = 1 - F$ of a distribution function F concentrated on the nonnegative half-line and its quantile function τ_q . Here, if F is continuous and increasing, then both properties of inverse functions hold, i.e., $\bar{F}(\tau_q) = q$ and $\tau_{\bar{F}(x)} = x$. On the other hand, if F is either discontinuous or non-increasing, then each of these may fail. Nevertheless, one can derive the asymptotic behavior of one of these dual objects, either $\bar{F}(x)$ as $x \rightarrow \infty$ or τ_q as $q \rightarrow 0$, from the other one.

A violation of any of the Assumptions (a)–(c) means in general that the duality between $\{N_t\}$ and $\{S_n\}$ disappears. Since duality properties are important in many situations, one main aim of this monograph is to study conditions under which duality retains, more precisely, under which an almost sure convergence of $S_n/a(n)$ as $n \rightarrow \infty$ results in an almost sure convergence of $N(t)/a^{-1}(t)$ as $t \rightarrow \infty$, and vice versa, where $a(\cdot)$ is a suitable normalizing function with inverse a^{-1} .

In Klesov et al. [227], for example, such dualities have been proved in a general setting and it became clear that this could be done not only for N_t defined by (1), but also for a large class of generalized renewal processes including the functionals L_t and T_t . The key observation in Klesov et al. [227] was that, in order to preserve duality, the inverse function a^{-1} should satisfy a technical condition, namely

$$\lim_{\varepsilon \downarrow 0} \limsup_{t \rightarrow \infty} \left| \frac{a^{-1}((1 \pm \varepsilon)t)}{a^{-1}(t)} - 1 \right| = 0. \quad (2)$$

Property (2) is satisfied, for example, if a is a *regularly varying* (RV) function (in the Karamata sense) with nonzero index.

Now, considering (2) as the defining property of a more general class \mathcal{PRV} of functions, called *pseudo-regularly varying* (PRV) functions, and developing its characteristics further, it turns out that the theory of dual objects can be extended. It will be shown that many properties of PRV functions remain the same as in the RV case, including, e.g., an integral representation and uniform convergence properties. Having then developed a theory of PRV functions one is able to study the duality of objects in a unified manner, under which the classical setting corresponds to a particular, still very important example.

The structure of the book has certainly been influenced by the direction we have taken in our investigations of PRV functions, namely, we first obtained some basic applications to classical limit theorems in renewal theory, then we understood the importance of the PRV property (2) in this field and discovered its central role for various other limit theorems, and finally we investigated a number of further

applications in different fields. Before we now briefly describe the contents of the book, we would like to mention that the asymptotics for generalized renewal processes studied here are essentially based on the fact that the objects under consideration are inverse to each other (in a certain sense), e.g., $S_{N_t} \approx t$ and $N(S_n) \approx n$, where “ \approx ” has to be given a precise meaning, of course.

We should mention that we have not aimed to touch *all* the interesting aspects and important applications of regular variation theory. Our scope is to add some new material to the various monographs and texts which are already devoted to these topics (see, for example, Bingham et al. [41] for a comprehensive discussion of both the theory and applications of regular variation). For the development of the theory see also the important contributions of Seneta [324] and Geluk and de Haan [148]. An excellent presentation of Tauberian theorems which is heavily related to regularly varying functions has been given in Korevaar [237]. Resnick [300, 302], de Haan and Ferreira [167], and Borovkov and Borovkov [48] discuss various applications of the concept of regular variation in probability theory. A role of regularly varying functions for actuarial and finance mathematics is highlighted in Embrechts et al. [119] and Novak [285]. For recent texts dealing with applications to statistical problems and long range dependence see, e.g., Mikosch [272], Samorodnitsky [316], Solier [340] or Pipiras and Taqqu [292] (see also the interesting discussion in de Haan [166], where applications to currency exchange rates, life span estimation, and sea level data are given).

In Chap. 1, we first assume the classical setting and study equivalences in the strong law of large numbers and the law of the iterated logarithm for sums of independent, identically distributed (iid) random variables and their corresponding renewal processes. As mentioned above, the proofs are essentially based on the property that the renewal process $N = \{N_t\}$ is the generalized inverse function constructed from the sequence $S = \{S_n\}$ of partial sums. Not only equivalence statements can be proved in the case of sums of iid random variables, but also the corresponding moment conditions can be derived from their counterparts for partial sums. Some nontraditional limit results for sums are also studied in this chapter.

Chapter 2 is a continuation of Chap. 1 in a more general setting, where much less is known about the limit properties of the underlying sequences. The random variables studied in this chapter are neither necessarily independent, nor identically distributed, nor nonnegative. We propose some new approaches to derive limit results for generalized renewal processes from their counterparts holding for the underlying sequences. Note that several definitions of generalized renewal processes are introduced in this chapter, where each of them reflects a certain feature of the classical definitions. Among the generalized renewal processes studied in Chap. 2 are the *first exit time*, *last exit time*, and *sojourn time*.

Chapters 3–7 provide the function-theoretic foundations of the book, while the other chapters are devoted to applications and some necessary complements. The following three classes of functions play a key role when studying dual objects in this monograph, namely \mathcal{PRV} , the class of *pseudo regularly varying* functions, \mathcal{SQJ} , the class of *sufficiently quickly increasing* functions, and \mathcal{POV} , the class of *pseudo-O-varying* functions. In fact, all three classes appear as natural generalizations of

the classical class of RV functions, but other important classes of functions will be considered as well.

In Chap. 4, special attention is paid to absolutely continuous functions (including paths of compound Poisson processes in particular). An interesting link to the property of *elasticity* in mathematical economics is established when studying absolutely continuous PRV functions.

The classes \mathcal{PRV} , \mathcal{SQJ} and \mathcal{POV} can be characterized in terms of *upper* and *lower limit functions* used in studying RV and (so-called) ORV functions, an extension of the notion of RV functions due to Avakumović and Karamata. A detailed treatment of functions belonging to the classes \mathcal{PRV} , \mathcal{SQJ} and \mathcal{POV} as well as their quasi-inverses is given in Chaps. 3, 4 and 7.

In Chap. 7, in particular, duality properties are discussed concerning the limit behavior of the ratio of two such functions and the corresponding ratio of their asymptotic quasi-inverses. The latter duality can be established via piecewise linear interpolation. We also study conditions, important in applications, under which generalized inverses are asymptotic inverse or quasi-inverse functions. Moreover, an application of the general results to ordinary differential equations is discussed, more precisely, it is studied how to obtain asymptotic stability of solutions of the Cauchy problem with respect to initial conditions.

The main scope of Chap. 5 is to study the class of ORV functions with *non-degenerate semigroups of regular points* introduced in Buldygin et al. [60]. This class contains the functions for which the limit $\lim_{t \rightarrow \infty} f(t\lambda)/f(t)$ exists and belongs to the interval $(0, \infty)$, but not necessarily for all numbers λ . It turns out that the set of such numbers λ is a multiplicative semigroup in \mathbf{R}_+ . All the functions discussed can be classified according to a given semigroup of regular points, in particular, RV functions correspond to the case where this semigroup coincides with \mathbf{R}_+ . Analogues of Karamata's theorem (both direct and inverse parts) are discussed in Chap. 6.

In Chaps. 7 and 8, we are able to generalize some results of Chap. 2, which turns out to also explain the key role of condition (2). In fact, it means that the function a^{-1} there belongs to the class \mathcal{SQJ} of sufficiently quickly increasing functions. It is worthwhile to mention that our approach provides a unified method to study the asymptotic behavior of renewal processes in both discrete and continuous time. The results of Chap. 8 reveal the nature of SQI, PRV, and POV properties of normalizing sequences and functions, and moreover contain conditions for SQI, PRV, and POV properties that can easily be checked. In addition to general results, Chap. 8 contains a number of applications to various specific schemes for constructing renewal processes. In particular, we prove a strong law of large numbers for generalized renewal processes constructed from nonhomogeneous compound Poisson processes.

Another application of the theory of POV functions is presented in Chap. 9. We obtain the exact order of growth (to infinity) of solutions of autonomous stochastic differential equations. The same method is applied to some other stochastic differential equations. The results of this chapter are rather general extensions of the corresponding theorems in the monograph of Gihman and Skorohod [149].

In Chap. 10, the method of Chap. 2 is also successfully applied to the case of renewal processes constructed from random walks in multidimensional time. Since there is no complete ordering in the space of multi-indices, the definition of a renewal process can only be justified by using an analogue of the process $\{T_t\}$ rather than $\{N_t\}$. The asymptotic behavior of renewal processes in this case differs from the classical one and depends on the asymptotic properties of the Dirichlet function, i.e., the number of solutions of the inequality $n_1 \cdots n_r \leq x$ for positive integers n_1, \dots, n_d , where x is a parameter. Some of our results require a more precise asymptotic for the Dirichlet function than those known so far. In fact, this behavior depends on a confirmation of the Riemann hypothesis concerning the zeros of the ζ -function.

The final Chap. 11 contains another application of PRV functions to the (so-called) precise asymptotics for the complete convergence of sums of stable random variables. Moreover, we give a new proof of Karamata's theorem that the function $W(t) = \sum_{s=1}^t w(s)/s$ is slowly varying if this holds true for w .

In a supplementary Appendix, we provide some auxiliary results, which may also be of independent interest and are partially new to the best of our knowledge.

We should finally mention that equivalent versions of the PRV property have been introduced by many authors and that PRV functions have been studied under various different names, e.g., in Korenblyum [235], Matuszewska [267], Matuszewska and Orlicz [270], Gihman and Skorohod [149], Stadtmüller and Trautner [344, 345], Berman [30, 32], Yakymiv [372], Cline [90], Djurčić [105], Djurčić and Torgašev [108], Klesov et al. [227], Buldygin et al. [61], and Rogozin [305], but the latter list is not exhaustive at all. The papers of Korenblyum [235] and Stadtmüller and Trautner [344] deal with nondecreasing PRV functions in the framework of Tauberian theorems for Laplace transforms. In particular, the Tauberian theorem for the Laplace transform of a nondecreasing positive function f holds if and only if $f \in \mathcal{PRV}$ (see [344]). A generalization of the PRV property to the multivariate case has been considered in Yakymiv [372] and, in fact, turns out to be important for the one-dimensional case as well (see also Resnick [301]).

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Acknowledgements

In a stimulating paper in 1930, Karamata [205] introduced the notion of *regular variation* and proved some fundamental theorems for *regularly varying* (RV) functions (see also Karamata [206]). These results together with later extensions and generalizations turned out to be very fruitful for various fields of mathematics (cf. Seneta [324] and Bingham et al. [41] for excellent surveys on this topic and for the history of its theory and applications). Among the classical generalizations there are, e.g., the (so-called) *O-regularly varying* (ORV) functions and *O-slowly varying* (OSV) functions. We further add to this list the notions of *pseudo-regularly varying* (PRV) functions and functions of *positive order of variation* (POV functions) together with other extensions which can effectively be applied in various mathematical problems.

Our interest in generalizing the notion of Karamata's regular variation was stimulated by some applications to certain asymptotic problems in renewal theory and can be traced back to the very first days of this century. Our starting point was condition (2.23) to be imposed on the required normalizing functions, whose complicated analytical form appeared unsatisfactory to us. So, in the beginning we just treated this condition as a purely technical one, being useful for deriving some equivalence theorems in renewal theory.

Later, we succeeded in improving it to a nicer form (see (3.10)) and we also learned that we were not the first to make use of this condition. In fact, it seems to be a root for several trees of studies of important classes of functions of a real variable.

Our approach to condition (3.10) has changed after an answer of Eugene Seneta to our question about his opinion and possible related results in the literature. He kindly provided us with a preliminary list of references and suggested to ask Tatjana Ostrogorski concerning further results on this topic. Unfortunately, we were not able to take advantage of a possible discussion of the topic with her due to her untimely death.

On the other hand, among other references in Prof. Seneta's list, we learned more about the origin of the topic, in particular of the work of Avakumović [19]. Vojislav Gregor Avakumović (1910–1990) was a late professor at the University of Marburg,

Germany, where some of his students and collaborators are still working. Due to this special coincidence of the topic and the place, where we were working on it, we decided to continue our investigations further. Thanks to the support given by the Deutsche Forschungsgemeinschaft (DFG) this could be realized.

Our interest in the generalization of Karamata's theory of regular variation has expanded over time and new settings and problems have come up ever since. We have worked enthusiastically on the topic and our knowledge of the field became wider and wider. However, we do not claim at all that it is comprehensive in any respect. Moreover, our continuous extension of the list of references indicates clearly that (very likely) some important results and authors are still missing.

The present book is a personal reflection of our joint research work on this subject, mainly done during numerous mutual visits of the authors to Germany and Ukraine. Our intention is to share our knowledge of the generalized Karamata theory and its various implications with all researchers and advanced students in the field. We hope it will be useful for some of them and it may contribute to further developing the theory and its potential applications. On the other hand, we clearly see that neither the theory, nor its applications, are in a final state yet, rather they are still in progress and expanding quickly. The structure of the book reflects the historical development of our approach, that is, first some applications are discussed, after which a basic theory is created, and finally further applications are provided.

We very much enjoyed and still appreciate our academic collaboration and take the opportunity to thank DFG and the Universities of Marburg, Paderborn, and Cologne in Germany and the National Technical University of Ukraine "KPI" in Kiev, Ukraine, for continuous support and help over the years.

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Let us work together in the beautiful field of Karamata's ideas and their extensions and applications!

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Contents

1	Equivalence of Limit Theorems for Sums of Random Variables and Renewal Processes	1
1.1	Introduction	1
1.2	Strong Limit Theorems for Partial Sums	3
1.3	Renewal Processes with Linear Drift	11
1.3.1	Results	11
1.3.2	Proofs	12
1.4	Renewal Processes Without Linear Drift	17
1.4.1	Results	17
1.4.2	Proofs	20
1.5	Comments	24
2	Almost Sure Convergence of Renewal Processes	27
2.1	Introduction	27
2.2	Almost Sure Limit Theorems	31
2.2.1	The Strong Law of Large Numbers	31
2.2.2	Non-equivalent Strong Laws	35
2.2.3	Rate of Convergence	38
2.3	Examples	43
2.3.1	Renewal Processes Constructed from Independent, Identically Distributed Random Variables	43
2.3.2	Nonidentically Distributed or Dependent Interarrival Times	48
2.4	Comments	51
3	Generalizations of Regularly Varying Functions	53
3.1	Introduction	53
3.2	RV- and ORV-Functions	54
3.2.1	Some Notation	54
3.2.2	Upper and Lower Limit Functions	56
3.2.3	RV- and SV-Functions	57

3.2.4	ORV- and OSV-Functions.....	59
3.2.5	OURV-Functions	60
3.3	Four Important Classes of Functions.....	61
3.3.1	PRV-Functions.....	61
3.3.1.1	Some Simple Properties of WPRV- and PRV-Functions	65
3.3.2	PI- and SQI-Functions	66
3.3.2.1	Some Simple Properties of WSQI- and SQI-Functions.....	68
3.3.3	POV-Functions	69
3.3.3.1	Some Simple Properties of POV-Functions ...	69
3.4	Functions Preserving the Asymptotic Equivalence.....	70
3.4.1	A Uniform Convergence Theorem	75
3.4.2	A Uniform Convergence Theorem for SV- and RV-Functions	75
3.5	Integral Representations	76
3.5.1	Integral Representations for RV- and ORV-Functions ...	76
3.5.2	Integral Representations for PRV-Functions	77
3.5.3	Integral Representations for POV-Functions.....	79
3.6	Potter's Bounds for PRV-Functions	80
3.7	Convergence to Infinity of PI-Functions	82
3.7.1	A Counterexample	89
3.8	Characterizations of POV-Functions	91
3.9	Asymptotically Equivalent Monotone Versions of POV-Functions	92
3.10	Comments	95
4	Properties of Absolutely Continuous Functions.....	99
4.1	Introduction	99
4.2	Some Classes of Absolutely Continuous Functions.....	100
4.3	Relationships Between Limit Functions	101
4.3.1	Essential Limit Functions	102
4.3.2	Inequalities Between the Limit Functions of f and Its Density θ	103
4.4	Logarithmic Density and Integral Representations.....	106
4.4.1	Integral Representations for Functions in the Class \mathbb{DL}	106
4.4.2	Logarithmic Densities.....	107
4.4.3	A Characterization of Properties of Absolutely Continuous Functions	107
4.4.4	Admissible Transformations of Densities	108
4.5	Elasticity and Indices of Asymptotic Elasticity	111
4.5.1	Elasticity of Functions	111
4.5.2	Indices of Asymptotic Elasticity.....	113
4.5.3	Essential Indices of Asymptotic Elasticity.....	114

4.5.4	Examples	115
4.5.5	Arithmetic Properties of Indices of Asymptotic Elasticity	118
4.6	Relationships in Terms of the Indices of Asymptotic Elasticity	119
4.7	Functions with Bounded Asymptotic Elasticity	121
4.7.1	Admissible Transformations of Densities	122
4.7.2	The ER-Property of Absolutely Continuous Functions	123
4.7.3	Boundaries of the Indices of Asymptotic Elasticity for Almost Monotone Densities	125
4.8	The PRV-Property of Absolutely Continuous Functions	130
4.8.1	Necessary PRV-Properties	130
4.8.2	The Measurability of Limit Functions of Densities and the ER- (PRV)-Property of Absolutely Continuous Functions	134
4.9	The SQI-Property of Absolutely Continuous Functions	135
4.9.1	Necessary Conditions for the PI-Property	136
4.9.2	Sufficient SQI-Conditions	138
4.10	The POV-Property of Absolutely Continuous Functions	139
4.11	Properties of Piecewise Linear Interpolations	141
4.11.1	Properties of Piecewise Linear Interpolations of Positive Sequences	141
4.11.2	Properties of Positive Piecewise Linear Functions	148
4.12	Comments	150
5	Nondegenerate Groups of Regular Points	153
5.1	Introduction	153
5.2	Groups of Regular Points	154
5.2.1	Regular Points and Groups of Regular Points	154
5.2.2	Full Sets of Regular Points for RV- (WRV)-Functions	156
5.2.3	Characterization Theorems	156
5.2.4	Null Sets of Regular Points	158
5.2.5	An Arbitrary Group of Regular Points	160
5.2.6	A Degenerate Group of Regular Points	161
5.3	Regularly Periodic and Regularly Log-Periodic Functions	161
5.3.1	Definitions and Some Properties of Regularly Periodic and Regularly Log-Periodic Functions	161
5.3.2	Sets of Regular Points for Regularly Log-Periodic Functions	166
5.4	*-Invariant Limit Functions	168
5.5	Weak Factorization Representations for Limit Functions	172
5.5.1	Some Corollaries and Remarks	174

5.6	Strong Factorization Representations for Limit Functions.....	175
5.6.1	Some Corollaries and Remarks	179
5.6.2	Factorization Representations of Limit Functions for ORV-Functions with Nondegenerate Sets of Regular Points	182
5.7	Characterization Theorems	182
5.8	Factorization Representations of Functions	186
5.9	Uniform Convergence Theorems.....	193
5.10	Comments	198
6	Karamata's Theorem for Integrals	201
6.1	Introduction	201
6.2	Karamata's Theorem for Integrals of RV-Functions.....	202
6.2.1	Karamata's Theorem: Direct Part	202
6.2.2	Karamata's Theorem: Converse Part	203
6.3	Regularly Log-Periodic and Regularly Log-Bounded Functions	203
6.3.1	Regularly Log-Bounded Functions.....	205
6.4	Generalizations of Karamata's Theorem.....	206
6.4.1	Direct Part: The Case $\rho \neq -1$	206
6.4.2	Direct Part: The Case $\rho = -1$	207
6.4.3	Converse Part	209
6.5	Asymptotic Properties of Integrals	210
6.6	An Auxiliary Result.....	215
6.7	Proof of the Generalized Karamata Theorem.....	219
6.7.1	Proof of the Converse Part of the Generalized Karamata Theorem	219
6.7.2	The Proof of the Generalized Karamata Theorem, Direct Part	226
6.8	Comments	227
7	Asymptotically Quasi-inverse Functions	229
7.1	Introduction	229
7.2	Asymptotically Quasi-inverse and Asymptotically Inverse Functions	232
7.2.1	Generalized Inverse Functions.....	233
7.2.2	Quasi-inverse Functions	233
7.2.3	Asymptotically Quasi-inverse and Asymptotically Inverse Functions	235
7.2.4	Asymptotically Quasi-inverse Functions for PRV-Functions.....	237
7.2.5	Asymptotically Inverse Functions for POV-Functions	241
7.2.6	Asymptotically Right and Asymptotically Left Quasi-inverse Functions	243
7.3	Properties of Asymptotically Quasi-inverse Functions	244

- 7.4 Asymptotically Quasi-inverse Functions and the POV-Property 249
- 7.5 Properties of a Ratio of Asymptotically Quasi-inverse Functions 254
 - 7.5.1 A General Approach to the Proof 255
 - 7.5.2 The Limit Behavior of the Ratio of Functions Which Are Asymptotically Quasi-inverses for SQI-Functions 256
 - 7.5.3 A Characterization of SQI- and PRV-Functions 259
 - 7.5.4 The Limit Behavior of the Ratio of Functions Which Are Asymptotically Quasi-inverse for POV-Functions 260
 - 7.5.5 Zero and Infinite Limit Points of the Ratio of Asymptotically Quasi-inverse Functions 264
 - 7.5.6 Zero and Infinite Limit Points of the Ratio of Functions Which Are Asymptotically Quasi-inverse for POV-Functions 266
 - 7.5.7 The Limit Behavior of the Ratio of Functions Which Are Asymptotically Quasi-inverse for RV-Functions 269
- 7.6 Limit Points of the Ratio of Asymptotically Quasi-inverse Functions 270
 - 7.6.1 Necessary Conditions for Relation (7.44) 275
- 7.7 Duality Between Certain Classes of Functions 276
 - 7.7.1 Some Auxiliary Results 276
 - 7.7.2 The Images of Some Classes of Functions Under the Transformations $f \mapsto f^\sim$ and $f^\sim \mapsto f$ 278
 - 7.7.3 Proof of Theorems 7.73–7.78 279
 - 7.7.4 Equivalence Theorems for Asymptotically Left Quasi-inverse Functions 286
- 7.8 Generalized Inverse Functions Corresponding to Sequences 287
 - 7.8.1 The Limit Behavior of the Ratio of Asymptotically Quasi-inverse Functions Constructed from Sequences 290
 - 7.8.2 Piecewise Linear Interpolations of Sequences and Functions 291
 - 7.8.3 Generalized Inverse Functions for Piecewise Linear Interpolations of Sequences 294
 - 7.8.4 The Limit Behavior of the Ratio of Asymptotically Quasi-inverse Functions for Sequences 295
- 7.9 Stability of Solutions of Differential Equations 301
 - 7.9.1 Solutions of the Cauchy Problem for an Autonomous Differential Equation 301

7.9.2	Asymptotic Stability of a Solution of the Cauchy Problem with Respect to the Initial Condition for an Autonomous Differential Equation	303
7.9.3	Asymptotic Stability of Solutions of the Cauchy Problem with Respect to the Initial Condition for a Non-autonomous Ordinary Differential Equation	307
7.10	Comments	308
8	Generalized Renewal Processes	311
8.1	Introduction	311
8.2	Generalized Renewal Processes Constructed from Random Sequences	314
8.2.1	The SLLN for Random Sequences and the Corresponding Renewal Processes	315
8.3	Examples	320
8.3.1	Generalized Renewal Processes Constructed from Stationary Sequences.....	320
8.3.2	The Asymptotic Behavior of the First Exit Time from a Domain with a Curvilinear Boundary	322
8.3.3	Generalized Renewal Processes Constructed from Markovian Gaussian Sequences Under Strong Dependence.....	324
8.3.4	Generalized Renewal Processes Constructed from Sums of Independent Random Variables.....	326
8.4	Renewal Processes Constructed from Stochastic Processes	327
8.4.1	The SLLN for Stochastic Processes and Their Corresponding Renewal Processes	329
8.4.2	The SLLN for Stochastic Processes and Their Corresponding Renewal Processes Under WPRV Normalizations	331
8.4.3	The SLLN for Stochastic Processes and Their Corresponding Renewal Processes Under POV Normalizations	332
8.4.4	The SLLN for Stochastic Processes and Their Corresponding Renewal Processes Under RV Normalizations	333
8.5	Further Examples	334
8.5.1	Generalized Renewal Processes Constructed from Compound Counting Processes	336
8.5.2	Generalized Renewal Processes Constructed from Compound Poisson Processes	338
8.5.3	Generalized Renewal Processes Constructed from Compound Poisson Processes with Piecewise Constant Intensity Functions	340
8.6	Comments	341

9 Asymptotic Behavior of Solutions of Stochastic Differential Equations 345

9.1 Introduction 345

9.1.1 Gihman and Skorohod’s Setting and Approach..... 347

9.1.2 Organization of the Current Chapter 349

9.2 Order of Growth of Solutions of Stochastic Differential Equations 350

9.2.1 Auxiliary Functions and Their Properties 351

9.2.2 Main Results..... 355

9.2.3 The Exact Order of Growth of the Process $G_a \circ \tilde{X}_b$ 356

9.2.4 Some Auxiliary Results 357

9.2.5 Proof of the Main Results..... 362

9.2.6 A Discussion of the Main Results 364

9.2.7 Asymptotic Stability of Solutions of Stochastic Differential Equations with Respect to Initial Conditions 365

9.3 Renewal Processes Constructed from Solutions of Stochastic Differential Equations 366

9.4 Order of Growth of Solutions of Autonomous Stochastic Differential Equations 369

9.4.1 Renewal Processes Constructed from Solutions of Autonomous Stochastic Differential Equations..... 373

9.5 φ -Order of Growth of Solutions of Autonomous Stochastic Differential Equations 373

9.6 $\varphi_{1,2}$ -Equivalence of Solutions of Autonomous Ordinary Differential Equations 378

9.6.1 Conditions for the $\varphi_{1,2}$ -Asymptotic Equivalence of Solutions of Ordinary Differential Equations in Terms of the Functions $G_k^{(\varphi_k)}$ 380

9.6.2 Conditions for the $\varphi_{1,2}$ -Asymptotic Equivalence of Solutions of Ordinary Differential Equations in Terms of the Functions $g_k^{(\varphi_k)}$ 381

9.6.3 An Application of Karamata’s Theorem 382

9.7 $\varphi_{1,2}$ -Order of Growth of Solutions of Autonomous Stochastic Differential Equations 384

9.8 $\varphi_{1,2}$ -Equivalence of Solutions of Autonomous Stochastic Differential Equations 387

9.9 Comments 391

10 Asymptotics for Renewal Processes Constructed from Multi-indexed Random Walks 395

10.1 Introduction 395

10.2 Some Classical Results from Renewal Theory 397

10.3 Limit Theorems for Multiple Sums 398

10.4 The Renewal Function for Multi-indexed Random Walks 400

- 10.5 Renewal Processes Constructed from Multi-indexed Random Walks 402
- 10.6 Asymptotics for Renewal Functions Constructed from Random Walks with Multidimensional Time 403
- 10.7 Asymptotics for Renewal Processes Constructed from Random Walks with Multidimensional Time 405
- 10.8 Examples 408
- 10.9 The Leading Coefficient in the Dirichlet Divisor Problem 411
- 10.10 An Example of Kátaı 414
- 10.11 Comments 416
- 11 Spitzer Series and Regularly Varying Functions** 419
 - 11.1 Introduction 419
 - 11.2 The Asymptotic Behavior of Large Deviation Probabilities 423
 - 11.3 Proof of Theorem 11.1 433
 - 11.4 Comments 438
- A Some Auxiliary Results** 439
 - A.1 Introduction 439
 - A.2 The Asymptotic Behavior of Series of Weighted Tails of Distributions 440
 - A.3 Large Deviations in the Case of Attraction to a Stable Law 443
 - A.4 Some Auxiliary Results on Slowly Varying Functions 449
 - A.5 Distributions Attracted to Stable Laws 452
 - A.6 The Asymptotic Behavior of Integrals and Sums with Slowly Varying Functions 458
 - A.7 The Parameswaran Lemma for Sums 459
 - A.8 Comments 460
- References** 463
- Index** 479