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Pseudo-Regularly Varying Functions and Generalized Renewal Processes

Springer

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Preface

Renewal theory is a branch of probability theory rich in fascinating mathematical problems and also in various important applications. On the other hand, regular variation of functions is a property that plays a key role in many fields of mathematics. One of the main aims of this book is to exhibit some fruitful links between these two areas via a generalized approach to both of them.

The core of renewal theory is to study (so-called) renewal processes and their probabilistic and statistical characteristics. One of the most cited examples in renewal theory is the following, which deals with the life span of a light bulb.

Renewal Process Assume one has a light bulb in a room and one turns it on, keeping the light bulb working until it fails, after which it is replaced with a new light bulb. If ξ_i denotes the life span of the ith bulb, then $S_n = \xi_1 + \cdots + \xi_n$ represents the total life span of the first n bulbs and

$$N_t = \max\{n : S_n \le t\} \tag{1}$$

is the number of bulbs needed until time t. $\{N_t\}$ is called a renewal counting process constructed from the sequence $\{S_n\}$.

Of course, the bulbs in this example can be exchanged by any other expendable resource, which makes the example more realistic and more attractive from the point of view of applications. For example, consider arriving customers waiting in a queue until one of the servers is free to serve him/her. The arrival counting process to the queue is usually assumed to be a renewal counting process $\{N_t\}$, where the $\{\xi_i\}$ are the inter-arrival times between the customers.

Next we give some further examples of stochastic processes which are related to renewal processes and which occur both in pure and applied mathematics.

Reward Process Let some (random) event occur from time to time. Imagine that someone experiences a reward at each occurrence of the event. Let r_i be the reward earned at the time of the ith occurrence of the event. Denoting by N_t the number of occurrences of the event up to time t, we again see a renewal process with ξ_i being the time between the $(i-1)$th and ith occurrence of the event. Then

$$Y_t = \sum_{i=1}^{N_t} r_i$$

is called a reward process, which is a special case of a (so-called) compound renewal process.

Treating the r_i's as penalties rather than rewards we arrive at several other settings. One of the classical examples here arises in a risk model describing the evolution of the capital of an insurance company which experiences two opposing cash flows: incoming cash premiums and outgoing claims. Premiums from the customers may arrive at a constant rate $c > 0$ (say) and claims $\{r_i\}$ occur according to a counting process $\{N_t\}$, that is, N_t is the number of claims up to time t. So, for an insurer who starts with initial surplus x,

$$X_t = x + ct - \sum_{i=1}^{N_t} r_i, \qquad t \geq 0,$$

represents the capital at time t.

General Counting Process There are many situations similar to what has been described above. Let, for example, ξ_i be the time passed between the $(i-1)$th and ith record in a sport discipline. Then N_t, again, denotes the number of records up to time t and is, in general, called a counting process. Records can also have a negative meaning, for example losses caused by catastrophic events such as hurricanes.

Such a counting process can also be seen in pure mathematics. Let (say) S_n be the nth prime number. Then N_t (usually denoted by $\pi(t)$ in this case) plays a crucial role not only in number theory, but also in applications like cryptography when estimating the security of protocols used to transmit data. An example where a renewal process explicitly occurs in cryptomachines is discussed in Gut [159].

The central object of all the above models can be reduced to an investigation of the behavior of an underlying counting process $\{N_t\}$. In probability theory, this can effectively be done under the classical assumptions that the $\{\xi_i\}$ are

(a) independent,
(b) identically distributed,
(c) positive

random variables. But what about the asymptotic behavior of N_t in cases where these assumptions fail? Such situations are not rare at all. Imagine, for example, that a catastrophic event occurs and leads to many claims for damages. Then the times $\{\xi_i\}$ between the claims become dependent, since many of the policy holders will report to an insurance company at (roughly) the same time. Another example, in which Assumption (b) fails, is related to the so-called alternating renewal process

$\{N_t\}$ constructed from a sequence

$$S_1 = \xi_1, \quad S_2 = \xi_1 + \eta_1, \quad S_3 = \xi_1 + \eta_1 + \xi_2, \quad S_4 = \xi_1 + \eta_1 + \xi_2 + \eta_2, \ldots,$$

where the $\{\xi_i\}$ are independent random variables with distribution function F_ξ and the $\{\eta_i\}$ are independent random variables with distribution function F_η. This kind of renewal process occurs in chromatographic models or in describing the motion of water in a river (see, e.g., Gut [159] for a short discussion and further references). Finally, Assumption (c) may fail if, for example, $\{\xi_i\}$ describes a money flow including both expenses and incomes, so that ξ_i could attain both positive and negative values.

Formula (1) nevertheless defines a process $\{N_t\}$, called a *generalized renewal process* in this case, even if one of the Assumptions (a)–(c) fails. Moreover, one can introduce and study some other important functionals of $\{S_n\}$ here. For example, viewing $N_t + 1$ defined via (1) as the *first time* when $\{S_n\}$ exceeds the level t, one can also consider L_t, the *last passage time* across the level t. There are other natural functionals, e.g., T_t, the total time spent by $\{S_n\}$ below the level t, etc. The just mentioned three functionals coincide if Assumption (c) holds, but otherwise they are different. All these functionals as well as many others are also called generalized renewal processes. So, generalized renewal processes arise either if N_t is constructed by (1), but one of the properties (a)–(c) fails, or if other functionals of $\{S_n\}$ (similar to N_t) are considered (we shall specify later what kind of functionals we have in mind when studying generalized renewal processes).

One of the basic questions studied in all models including renewal processes concerns the asymptotic behavior of N_t as $t \to \infty$ which, in turn, is determined by the asymptotic behavior of S_n as $n \to \infty$ (we shall clarify what we mean by "asymptotic behavior" later in this book). Therefore one may expect that the asymptotic properties of the models for (generalized) renewal processes can be derived from the asymptotic behavior of $\{S_n\}$. In turn, it is also to be expected that one can argue vice versa, i.e., that the asymptotic behavior of $\{S_n\}$ is determined by that of $\{N_t\}$. In this case, one would be able to derive statistical properties of the ξ_i's by observing the counting process $\{N_t\}$. So, $\{S_n\}$ and $\{N_t\}$ may be viewed as "dual objects" in a certain sense.

Generally, we call two objects *dual* if their asymptotic properties are related to each other as indicated above, that is, if a limit result for the first object implies a corresponding one for the second object, and vice versa. The duality of $\{N_t\}$ and $\{S_n\}$ has been proved in Gut et al. [160] for the classical setting where all the Assumptions (a)–(c) hold. We would like to mention that Doob [111] predicted this duality property in 1948, but until very recently the limit theorems for renewal processes and those for sums of random variables developed independently.

Counting processes and their dual (or "renewal") processes are often observed in number theory. For example, the total number $\pi(x)$ of prime numbers up to x and the n-th prime number p_n are dual objects, since $\pi(p_n) = n$. This duality is reflected by the *prime number theorem* stating that $\pi(x) \sim x/\ln(x)$ as $x \to \infty$ and the asymptotic behavior $p_n \sim n\ln(n)$ as $n \to \infty$; note that $n\ln(n)$ is inverse to $x/\ln(x)$ in a certain sense.

Probability theory provides other examples of dual objects. One of them is the number μ_n of records until time n in a sequence of random variables and the magnitude τ_n of the n-th record. Obviously $\mu_{\tau_n} = n$ and this duality allows one to study the asymptotic behavior of μ_n and τ_n simultaneously. By the way, the second property of inverse functions fails in both cases since, in general, $\tau_{\mu_n} \neq n$ and $\tau_{\pi(n)} \neq n$.

Another example is the duality between the tail $\overline{F} = 1 - F$ of a distribution function F concentrated on the nonnegative half-line and its quantile function τ_q. Here, if F is continuous and increasing, then both properties of inverse functions hold, i.e., $\overline{F}(\tau_q) = q$ and $\tau_{\overline{F}(x)} = x$. On the other hand, if F is either discontinuous or non-increasing, then each of these may fail. Nevertheless, one can derive the asymptotic behavior of one of these dual objects, either $\overline{F}(x)$ as $x \to \infty$ or τ_q as $q \to 0$, from the other one.

A violation of any of the Assumptions (a)–(c) means in general that the duality between $\{N_t\}$ and $\{S_n\}$ disappears. Since duality properties are important in many situations, one main aim of this monograph is to study conditions under which duality retains, more precisely, under which an almost sure convergence of $S_n/a(n)$ as $n \to \infty$ results in an almost sure convergence of $N(t)/a^{-1}(t)$ as $t \to \infty$, and vice versa, where $a(\cdot)$ is a suitable normalizing function with inverse a^{-1}.

In Klesov et al. [227], for example, such dualities have been proved in a general setting and it became clear that this could be done not only for N_t defined by (1), but also for a large class of generalized renewal processes including the functionals L_t and T_t. The key observation in Klesov et al. [227] was that, in order to preserve duality, the inverse function a^{-1} should satisfy a technical condition, namely

$$\lim_{\varepsilon \downarrow 0} \limsup_{t \to \infty} \left| \frac{a^{-1}((1 \pm \varepsilon)t)}{a^{-1}(t)} - 1 \right| = 0. \tag{2}$$

Property (2) is satisfied, for example, if a is a *regularly varying* (RV) function (in the Karamata sense) with nonzero index.

Now, considering (2) as the defining property of a more general class $\mathcal{PRV}$ of functions, called *pseudo-regularly varying* (PRV) functions, and developing its characteristics further, it turns out that the theory of dual objects can be extended. It will be shown that many properties of PRV functions remain the same as in the RV case, including, e.g., an integral representation and uniform convergence properties. Having then developed a theory of PRV functions one is able to study the duality of objects in a unified manner, under which the classical setting corresponds to a particular, still very important example.

The structure of the book has certainly been influenced by the direction we have taken in our investigations of PRV functions, namely, we first obtained some basic applications to classical limit theorems in renewal theory, then we understood the importance of the PRV property (2) in this field and discovered its central role for various other limit theorems, and finally we investigated a number of further applications in different fields. Before we now briefly describe the contents of the book, we would like to mention that the asymptotics for generalized renewal processes studied here are essentially based on the fact that the objects under consideration are inverse to each other (in a certain sense), e.g., $S_{N_t} \approx t$ and $N(S_n) \approx n$, where "$\approx$" has to be given a precise meaning, of course.

We should mention that we have not aimed to touch *all* the interesting aspects and important applications of regular variation theory. Our scope is to add some new material to the various monographs and texts which are already devoted to these topics (see, for example, Bingham et al. [41] for a comprehensive discussion of both the theory and applications of regular variation). For the development of the theory see also the important contributions of Seneta [324] and Geluk and de Haan [148]. An excellent presentation of Tauberian theorems which is heavily related to regularly varying functions has been given in Korevaar [237]. Resnick [300, 302], de Haan and Ferreira [167], and Borovkov and Borovkov [48] discuss various applications of the concept of regular variation in probability theory. A role of regularly varying functions for actuarial and finance mathematics is highlighted in Embrechts et al. [119] and Novak [285]. For recent texts dealing with applications to statistical problems and long range dependence see, e.g., Mikosch [272], Samorodnitsky [316], Solier [340] or Pipiras and Taqqu [292] (see also the interesting discussion in de Haan [166], where applications to currency exchange rates, life span estimation, and sea level data are given).

In Chap. 1, we first assume the classical setting and study equivalences in the strong law of large numbers and the law of the iterated logarithm for sums of independent, identically distributed (iid) random variables and their corresponding renewal processes. As mentioned above, the proofs are essentially based on the property that the renewal process $N = \{N_t\}$ is the generalized inverse function constructed from the sequence $S = \{S_n\}$ of partial sums. Not only equivalence statements can be proved in the case of sums of iid random variables, but also the corresponding moment conditions can be derived from their counterparts for partial sums. Some nontraditional limit results for sums are also studied in this chapter.

Chapter 2 is a continuation of Chap. 1 in a more general setting, where much less is known about the limit properties of the underlying sequences. The random variables studied in this chapter are neither necessarily independent, nor identically distributed, nor nonnegative. We propose some new approaches to derive limit results for generalized renewal processes from their counterparts holding for the underlying sequences. Note that several definitions of generalized renewal processes are introduced in this chapter, where each of them reflects a certain feature of the classical definitions. Among the generalized renewal processes studied in Chap. 2 are the *first exit time*, *last exit time*, and *sojourn time*.

Chapters 3–7 provide the function-theoretic foundations of the book, while the other chapters are devoted to applications and some necessary complements. The following three classes of functions play a key role when studying dual objects in this monograph, namely $\mathcal{PRV}$, the class of *pseudo regularly varying* functions, $\mathcal{SQI}$, the class of *sufficiently quickly increasing* functions, and $\mathcal{POV}$, the class of *pseudo-O-varying* functions. In fact, all three classes appear as natural generalizations of the classical class of RV functions, but other important classes of functions will be considered as well.

In Chap. 4, special attention is paid to absolutely continuous functions (including paths of compound Poisson processes in particular). An interesting link to the property of *elasticity* in mathematical economics is established when studying absolutely continuous PRV functions.

The classes $\mathcal{PRV}$, $\mathcal{SQI}$ and $\mathcal{POV}$ can be characterized in terms of *upper* and *lower limit functions* used in studying RV and (so-called) ORV functions, an extension of the notion of RV functions due to Avakumović and Karamata. A detailed treatment of functions belonging to the classes $\mathcal{PRV}$, $\mathcal{SQI}$ and $\mathcal{POV}$ as well as their quasi-inverses is given in Chaps. 3, 4 and 7.

In Chap. 7, in particular, duality properties are discussed concerning the limit behavior of the ratio of two such functions and the corresponding ratio of their asymptotic quasi-inverses. The latter duality can be established via piecewise linear interpolation. We also study conditions, important in applications, under which generalized inverses are asymptotic inverse or quasi-inverse functions. Moreover, an application of the general results to ordinary differential equations is discussed, more precisely, it is studied how to obtain asymptotic stability of solutions of the Cauchy problem with respect to initial conditions.

The main scope of Chap. 5 is to study the class of ORV functions with *non-degenerate semigroups of regular points* introduced in Buldygin et al. [60]. This class contains the functions for which the limit $\lim_{t\to\infty} f(t\lambda)/f(t)$ exists and belongs to the interval $(0,\infty)$, but not necessarily for all numbers λ. It turns out that the set of such numbers λ is a multiplicative semigroup in $\mathbf{R}_+$. All the functions discussed can be classified according to a given semigroup of regular points, in particular, RV functions correspond to the case where this semigroup coincides with $\mathbf{R}_+$. Analogues of Karamata's theorem (both direct and inverse parts) are discussed in Chap. 6.

In Chaps. 7 and 8, we are able to generalize some results of Chap. 2, which turns out to also explain the key role of condition (2). In fact, it means that the function a^{-1} there belongs to the class $\mathcal{SQI}$ of sufficiently quickly increasing functions. It is worthwhile to mention that our approach provides a unified method to study the asymptotic behavior of renewal processes in both discrete and continuous time. The results of Chap. 8 reveal the nature of SQI, PRV, and POV properties of normalizing sequences and functions, and moreover contain conditions for SQI, PRV, and POV properties that can easily be checked. In addition to general results, Chap. 8 contains a number of applications to various specific schemes for constructing renewal processes. In particular, we prove a strong law of large numbers for generalized renewal processes constructed from nonhomogeneous compound Poisson processes.

Another application of the theory of POV functions is presented in Chap. 9. We obtain the exact order of growth (to infinity) of solutions of autonomous stochastic differential equations. The same method is applied to some other stochastic differential equations. The results of this chapter are rather general extensions of the corresponding theorems in the monograph of Gihman and Skorohod [149].

In Chap. 10, the method of Chap. 2 is also successfully applied to the case of renewal processes constructed from random walks in multidimensional time. Since there is no complete ordering in the space of multi-indices, the definition of a renewal process can only be justified by using an analogue of the process $\{T_t\}$ rather than $\{N_t\}$. The asymptotic behavior of renewal processes in this case differs from the classical one and depends on the asymptotic properties of the Dirichlet function, i.e., the number of solutions of the inequality $n_1 \cdots n_r \leq x$ for positive integers $n_1, \ldots, n_d$, where x is a parameter. Some of our results require a more precise asymptotic for the Dirichlet function than those known so far. In fact, this behavior depends on a confirmation of the Riemann hypothesis concerning the zeros of the ζ-function.

The final Chap. 11 contains another application of PRV functions to the (so-called) precise asymptotics for the complete convergence of sums of stable random variables. Moreover, we give a new proof of Karamata's theorem that the function $W(t) = \sum_{s=1}^{t} w(s)/s$ is slowly varying if this holds true for w.

In a supplementary Appendix, we provide some auxiliary results, which may also be of independent interest and are partially new to the best of our knowledge.

We should finally mention that equivalent versions of the PRV property have been introduced by many authors and that PRV functions have been studied under various different names, e.g., in Korenblyum [235], Matuszewska [267], Matuszewska and Orlicz [270], Gihman and Skorohod [149], Stadtmüller and Trautner [344, 345], Berman [30, 32], Yakymiv [372], Cline [90], Djurčić [105], Djurčić and Torgašev [108], Klesov et al. [227], Buldygin et al. [61], and Rogozin [305], but the latter list is not exhaustive at all. The papers of Korenblyum [235] and Stadtmüller and Trautner [344] deal with nondecreasing PRV functions in the framework of Tauberian theorems for Laplace transforms. In particular, the Tauberian theorem for the Laplace transform of a nondecreasing positive function f holds if and only if $f \in \mathcal{PRV}$ (see [344]). A generalization of the PRV property to the multivariate case has been considered in Yakymiv [372] and, in fact, turns out to be important for the one-dimensional case as well (see also Resnick [301]).

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In a stimulating paper in 1930, Karamata [205] introduced the notion of *regular variation* and proved some fundamental theorems for *regularly varying* (RV) functions (see also Karamata [206]). These results together with later extensions and generalizations turned out to be very fruitful for various fields of mathematics (cf. Seneta [324] and Bingham et al. [41] for excellent surveys on this topic and for the history of its theory and applications). Among the classical generalizations there are, e.g., the (so-called) *O-regularly varying* (ORV) functions and *O-slowly varying* (OSV) functions. We further add to this list the notions of *pseudo-regularly varying* (PRV) functions and functions of *positive order of variation* (POV functions) together with other extensions which can effectively be applied in various mathematical problems.

Our interest in generalizing the notion of Karamata's regular variation was stimulated by some applications to certain asymptotic problems in renewal theory and can be traced back to the very first days of this century. Our starting point was condition (2.23) to be imposed on the required normalizing functions, whose complicated analytical form appeared unsatisfactory to us. So, in the beginning we just treated this condition as a purely technical one, being useful for deriving some equivalence theorems in renewal theory.

Later, we succeeded in improving it to a nicer form (see (3.10)) and we also learned that we were not the first to make use of this condition. In fact, it seems to be a root for several trees of studies of important classes of functions of a real variable.

Our approach to condition (3.10) has changed after an answer of Eugene Seneta to our question about his opinion and possible related results in the literature. He kindly provided us with a preliminary list of references and suggested to ask Tatjana Ostrogorski concerning further results on this topic. Unfortunately, we were not able to take advantage of a possible discussion of the topic with her due to her untimely death.

On the other hand, among other references in Prof. Seneta's list, we learned more about the origin of the topic, in particular of the work of Avakumović [19]. Vojislav Gregor Avakumović (1910–1990) was a late professor at the University of Marburg,

Germany, where some of his students and collaborators are still working. Due to this special coincidence of the topic and the place, where we were working on it, we decided to continue our investigations further. Thanks to the support given by the Deutsche Forschungsgemeinschaft (DFG) this could be realized.

Our interest in the generalization of Karamata's theory of regular variation has expanded over time and new settings and problems have come up ever since. We have worked enthusiastically on the topic and our knowledge of the field became wider and wider. However, we do not claim at all that it is comprehensive in any respect. Moreover, our continuous extension of the list of references indicates clearly that (very likely) some important results and authors are still missing.

The present book is a personal reflection of our joint research work on this subject, mainly done during numerous mutual visits of the authors to Germany and Ukraine. Our intention is to share our knowledge of the generalized Karamata theory and its various implications with all researchers and advanced students in the field. We hope it will be useful for some of them and it may contribute to further developing the theory and its potential applications. On the other hand, we clearly see that neither the theory, nor its applications, are in a final state yet, rather they are still in progress and expanding quickly. The structure of the book reflects the historical development of our approach, that is, first some applications are discussed, after which a basic theory is created, and finally further applications are provided.

We very much enjoyed and still appreciate our academic collaboration and take the opportunity to thank DFG and the Universities of Marburg, Paderborn, and Cologne in Germany and the National Technical University of Ukraine "KPI" in Kiev, Ukraine, for continuous support and help over the years.

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Let us work together in the beautiful field of Karamata's ideas and their extensions and applications!

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