

SpringerBriefs in Mathematics

Series editors

Nicola Bellomo, Torino, Italy
Michele Benzi, Atlanta, USA
Palle Jorgensen, Iowa City, USA
Tatsien Li, Shanghai, China
Roderick Melnik, Waterloo, Canada
Lothar Reichel, Kent, USA
Otmar Scherzer, Vienna, Austria
Benjamin Steinberg, New York, USA
Yuri Tschinkel, New York, USA
George Yin, Detroit, USA
Ping Zhang, Kalamazoo, USA

SpringerBriefs in Mathematics showcases expositions in all areas of mathematics and applied mathematics. Manuscripts presenting new results or a single new result in a classical field, new field, or an emerging topic, applications, or bridges between new results and already published works, are encouraged. The series is intended for mathematicians and applied mathematicians.

BCAM SpringerBriefs

Editorial Board

Enrique Zuazua

Departamento de Matemáticas
Universidad Autónoma de Madrid
Cantoblanco, Madrid, Spain

Irene Fonseca

Center for Nonlinear Analysis
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, USA

Juan J. Manfredi

Department of Mathematics
University of Pittsburgh
Pittsburgh, USA

Emmanuel Trélat

Laboratoire Jacques-Louis Lions
Institut Universitaire de France
Université Pierre et Marie Curie
CNRS, UMR, Paris

Xu Zhang

School of Mathematics
Sichuan University
Chengdu, China

BCAM *SpringerBriefs* aims to publish contributions in the following disciplines: Applied Mathematics, Finance, Statistics and Computer Science. BCAM has appointed an Editorial Board, who evaluate and review proposals.

Typical topics include: a timely report of state-of-the-art analytical techniques, bridge between new research results published in journal articles and a contextual literature review, a snapshot of a hot or emerging topic, a presentation of core concepts that students must understand in order to make independent contributions.

Please submit your proposal to the Editorial Board or to Francesca Bonadei, Executive Editor Mathematics, Statistics, and Engineering: francesca.bonadei@springer.com



John Guaschi · Daniel Juan-Pineda
Silvia Millán López

The Lower Algebraic
K-Theory of Virtually Cyclic
Subgroups of the Braid
Groups of the Sphere
and of $\mathbb{Z}[B_4(S^2)]$

John Guaschi
Laboratoire de Mathématiques Nicolas
Oresme UMR CNRS 6139
Université de Caen Normandie,
Normandie Université
Caen, France

Silvia Millán López
Colegio de Bachilleras del Estado de
Tlaxcala
Tlaxcala, Mexico

Daniel Juan-Pineda
Centro de Ciencias Matemáticas
Universidad Nacional Autónoma de México
Morelia, Michoacán, Mexico

ISSN 2191-8198 ISSN 2191-8201 (electronic)
SpringerBriefs in Mathematics
ISBN 978-3-319-99488-8 ISBN 978-3-319-99489-5 (eBook)
<https://doi.org/10.1007/978-3-319-99489-5>

Library of Congress Control Number: 2018952597

Mathematics Subject Classification (2010): 20F36, 19A31, 19B28, 14C35, 18F25, 13D15, 16S34, 20C05, 20E45, 20G05

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2018

This work is subject to copyright. All rights are solely and exclusively licensed by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

We study K -theoretical aspects of the braid groups $B_n(\mathbb{S}^2)$ on n strings of the 2-sphere, which by results of the second two authors are known to satisfy the Farrell–Jones fibred isomorphism conjecture [1]. In light of this, in order to determine the algebraic K -theory of the group ring $\mathbb{Z}[B_n(\mathbb{S}^2)]$, one should first compute that of its virtually cyclic subgroups, which were classified by D. L. Gonçalves and the first author [2]. We calculate the Whitehead and K_{-1} -groups of the group rings of the finite subgroups (dicyclic and binary polyhedral) of $B_n(\mathbb{S}^2)$ for all $4 \leq n \leq 11$. Some new phenomena occur, such as the appearance of torsion for the K_{-1} -groups. We then go on to study the case $n = 4$ in detail, which is the smallest value of n for which $B_n(\mathbb{S}^2)$ is infinite. We show that $B_4(\mathbb{S}^2)$ is an amalgamated product of two finite groups, from which we are able to determine a universal space for proper actions of the group $B_4(\mathbb{S}^2)$. We also calculate the algebraic K -theory of the infinite virtually cyclic subgroups of $B_4(\mathbb{S}^2)$, including the Nil groups of the quaternion group of order 8. This enables us to determine the lower algebraic K -theory of $\mathbb{Z}[B_4(\mathbb{S}^2)]$.

Caen, France
Morelia, Mexico
Tlaxcala, Mexico

John Guaschi
Daniel Juan-Pineda
Silvia Millán López

References

1. D. Juan-Pineda, S. Millán-López, The Whitehead group and the lower algebraic K -theory of braid groups on \mathbb{S}^2 and $\mathbb{R}P^2$. *Algebr. Geom. Topol.* **10**, 1887–1903 (2010)
2. D. L. Gonçalves, J. Guaschi, The virtually cyclic subgroups of the braid groups of the sphere, *Springer Briefs in Mathematics*, Springer (2013)

Acknowledgements

We wish to thank the following colleagues for helpful and fruitful discussions: Bruno Anglès, for help with the Galois theory of Sect. 2.5.2; Eric Jespers, Gerardo Raggi and Ángel del Río (and the GAP package *Wedderga* [1]), for aiding us with the Wedderburn decomposition of dicyclic and binary polyhedral groups; Jean-François Lafont, Ivonne Ortiz and Stratos Prassidis, for conversations on K -theoretical aspects of our work during the early stages of the writing of this paper; and Chuck Weibel, for valuable comments at various points. We would also like to thank the referees for their comments on the manuscript, and one referee in particular, who made a number of useful remarks and suggestions to make it easier to read.

The authors are grateful to the French-Mexican Laboratoire International Associé ‘Laisla’ for partial financial support. D. Juan-Pineda was partially supported by the CNRS, PAPIIT-UNAM and CONACYT (México), and J. Guaschi was partially supported by the ANR project TheoGar no ANR-08-BLAN-0269-02. J. Guaschi would like to thank the CNRS for having granted him a ‘délégation’ during the writing of part of this paper. J. Guaschi and S. Millán also wish to thank CONACYT (México) for partial financial support through its programme ‘Estancias postdoctorales y sabáticas vinculadas al fortalecimiento de la calidad del posgrado nacional’.

Reference

1. O. Broche Cristo, A. Konovalov, A. Olivieri, G. Olteanu, Á. del Río, *Wedderga–Wedderburn decomposition of group algebras*, GAP package, Version 4.3.3, (2009). <http://www.um.es/adelrio/wedderga.htm>.

Contents

1	Introduction	1
	References	4
2	Lower Algebraic K-Theory of the Finite Subgroups of $B_n(\mathbb{S}^2)$	7
2.1	Classification of the Virtually Cyclic Subgroups of $B_n(\mathbb{S}^2)$	7
2.2	Conjugacy Classes of Binary Polyhedral Groups	9
2.3	Whitehead Groups of the Finite Subgroups of $B_n(\mathbb{S}^2)$	13
2.4	$\tilde{K}_0(\mathbb{Z}[G])$ for the Finite Subgroups of $B_n(\mathbb{S}^2)$	14
2.5	$K_{-1}(\mathbb{Z}[G])$ for the Finite Subgroups of $B_n(\mathbb{S}^2)$	18
2.5.1	Torsion of $K_{-1}(\mathbb{Z}[G])$ for Finite Subgroups of $B_n(\mathbb{S}^2)$	18
2.5.2	The Rank of $K_{-1}(\mathbb{Z}[G])$ for the Finite Subgroups of $B_n(\mathbb{S}^2)$	24
2.6	The Lower Algebraic K -Theory of the Finite Subgroups of $B_n(\mathbb{S}^2)$ for $4 \leq n \leq 11$	39
	References	41
3	The Braid Group $B_4(\mathbb{S}^2)$, and the Conjugacy Classes of Its Maximal Virtually Cyclic Subgroups	43
3.1	Generalities About $B_4(\mathbb{S}^2)$	43
3.2	Maximal Virtually Cyclic Subgroups of $B_4(\mathbb{S}^2)$	47
3.2.1	Proof of Parts (a) and (b) of Theorem 41	48
3.2.2	Proof of Parts (c) and (d) of Theorem 41	50
3.2.3	Proof of the Existence of Maximal Subgroups $\mathcal{Q}_8 \times \mathbb{Z}$ in Part (c) of Theorem 41	55
3.3	Conjugacy Classes of Maximal Infinite Virtually Cyclic Subgroups in $B_4(\mathbb{S}^2)$	60
	References	62

- 4 Lower Algebraic K -Theory Groups of the Group**
- Ring $\mathbb{Z}[B_4(\mathbb{S}^2)]$** 63
- 4.1 The Lower K -Theory of Infinite Virtually Cyclic Groups 64
- 4.2 Preliminary K -Theoretical Calculations for $\mathbb{Z}[B_4(\mathbb{S}^2)]$ 65
- 4.3 Nil Group Computations 66
- References 71
- Appendix A: The Fibred Isomorphism Conjecture** 73
- Appendix B: Braid Groups** 75