

Research in Mathematics Education

Series editors

Jinfa Cai, Newark, DE, USA

James A. Middleton, Tempe, AZ, USA

This series is designed to produce thematic volumes, allowing researchers to access numerous studies on a theme in a single, peer-reviewed source. Our intent for this series is to publish the latest research in the field in a timely fashion. This design is particularly geared toward highlighting the work of promising graduate students and junior faculty working in conjunction with senior scholars. The audience for this monograph series consists of those in the intersection between researchers and mathematics education leaders—people who need the highest quality research, methodological rigor, and potentially transformative implications ready at hand to help them make decisions regarding the improvement of teaching, learning, policy, and practice. With this vision, our mission of this book series is:

1. To support the sharing of critical research findings among members of the mathematics education community;
2. To support graduate students and junior faculty and induct them into the research community by pairing them with senior faculty in the production of the highest quality peer-reviewed research papers; and
3. To support the usefulness and widespread adoption of research-based innovation.

More information about this series at <http://www.springer.com/series/13030>

Kelly S. Mix • Michael T. Battista
Editors

Visualizing Mathematics

The Role of Spatial Reasoning
in Mathematical Thought

 Springer

Editors

Kelly S. Mix
Department of Human Development
and Quantitative Methodology
University of Maryland
College Park, MD, USA

Michael T. Battista
Department of Teaching and Learning
The Ohio State University
Columbus, OH, USA

ISSN 2570-4729

ISSN 2570-4737 (electronic)

Research in Mathematics Education

ISBN 978-3-319-98766-8

ISBN 978-3-319-98767-5 (eBook)

<https://doi.org/10.1007/978-3-319-98767-5>

Library of Congress Control Number: 2018958712

© Springer Nature Switzerland AG 2018, corrected publication 2019

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors, and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Foreword

In 2016, we were approached by series editor, Dr. Jinfa Cai, with a novel idea—invite authors from the fields of developmental psychology and mathematics education to write about their work on spatial visualization and mathematics, and then ask them to write commentaries on one another’s chapters. The goal was to provide a unique view of research on this topic that encompassed both disciplines, as well as foster cross-field communication and intellectual synergy. We eagerly took up the challenge and invited scholars whose work we knew to be at the forefront of our respective fields. The chapters and commentaries contained in this volume are the products of this esteemed group. They reflect the state of the art in research on spatial visualization and mathematics from at least two perspectives. They highlight important new contributions, but they also reveal the fault lines between our respective disciplines. The commentaries insightfully point out some of these fault lines, as well as the immense common ground and the potential for deeper collaboration in the future.

The basic question of how spatial skill relates to mathematics has received steady attention over the years. In psychology, most of this work has focused on long-term outcomes in STEM fields for individuals with more advanced spatial skill (e.g., Wai, Lubinski, & Benbow, 2009), the possibility that spatial deficits contribute to poor mathematics outcomes in children (e.g., Geary, Hoard, Byrd-Craven, Nugent & Numtee, 2007), and the use of materials that physically embody (via spatial relations) abstract mathematics concepts (see Mix, 2010, for a review). Running through these disparate research programs is the shared notion that spatial thinking plays a major role in understanding mathematics, but it has not been addressed head on in psychology until recent years.

In mathematics education, Clements and Battista, in their 1992 research review, address just this issue. They wrote that both Hadamard and Einstein (renown mathematicians) claimed that much of the thinking required in higher mathematics is spatial, and they cited positive correlations between spatial ability and mathematics achievement at all grade levels. However, even in that time period, the relations between spatial thinking and learning nongeometric concepts did not seem straightforward, and there were conflicting findings. For some tasks, having high-spatial

skill seemed to improve performance, whereas in other tasks, processing mathematical information using verbal-logical reasoning enhanced performance compared to students who processed the information visually. Other mathematics education researchers countered that the understanding of some low-spatial students who did well in mathematics was instrumental, whereas high-spatial students' understanding was more relational, a difference often not captured by classroom or standardized assessments. Clements and Battista concluded that even though there was reason to believe that spatial reasoning is important in students' learning and use of mathematical concepts—including nongeometric concepts—the role that such reasoning plays in this learning remained elusive.

Possibly because of this elusiveness, interest in the topic waned in mathematics education. However, currently there is intense interest in this general topic in both psychology and mathematics education due to its potential educational benefits (Newcombe, 2010) and the insights into the relations found through extensive and detailed student interviews (Bruce et al., 2017; Davis et al., 2015). The chapters contributed to this volume represent various approaches to advancing this work in education or moving the work in both fields toward educational application.

The developmental psychology chapters tended to focus on the underlying mental representations used to understand mathematics, and the extent to which these representations already involve, or could be improved by spatial processing. Cipora, Schroeder, Soltanlou, and Nuerk provide a detailed analysis of the link between spatial and numerical processing purportedly demonstrated by spatial-numerical association (SNA) or mental number line effects. They conclude that spatial skills provide a crucial tool for understanding mathematics, but this relation may not be realized in the form of a fixed mental number line. Congdon, Vasileyva, Mix, and Levine examine a deep psychological structure that may underlie a range of mathematics topics—namely, the structure involved in identifying and enumerating spatial units of measurement. They argue that mastery of this structure has the potential to support mathematics learning throughout the elementary grades and perhaps head off misconceptions related to fractions, proportions, and conventional later on. Similarly, Jirout and Newcombe focus on another spatial relation with strong ties to mathematics—namely, relative magnitude—outline its potential role in improving instruction on whole number ordering, fractions, and proportions. Casey and Fell discuss the difference between general spatial skill and spatial skill instantiated in specific mathematics problems, concluding that the most effective way to leverage spatial training to improve mathematics outcomes is likely the latter. They highlight a number of instructional techniques from existing curricula that successfully use spatial representations. Finally, Young, Levine, and Mix considered the multidimensional nature of spatial processing and mathematics processing and the inherent complexity involved in identifying possible instructional levers. Following a critique of the existing literature, including recent factor analytic approaches, they conclude with a set of recommendations for improving these approaches and applying what is already known in educational settings.

The mathematics education chapters discuss the spatial processes involved in specific topics in mathematics. Sinclair, Moss, Hawes, and Stephenson examine

how children can learn “*through and from* drawing,” focusing on spatial processes and concepts in primary school geometry. They argue that drawing is not innate but can be improved, and they illustrate through fine-grained analysis how the potential benefits of geometric drawing can be realized in classrooms. Gutiérrez, Ramírez, Benedicto, Beltrán-Meneu, and Jaime analyze the spatial reasoning of mathematically gifted secondary school students as they worked on a collaborative, communication-intensive, task in which they were shown orthogonal projections of cube buildings along with related verbal information. The authors related the objectives of students’ actions and their visualization processes and students’ solution strategies and cognitive demand. Herbst and Boileau argue that high school geometry instruction can do more than provide names for 3D shapes and formulas for finding surface area and volume. They illustrate, and invite reflection on their design of, a 3D geometry modeling activity in which students write and interpret instructions for how to move pieces of furniture up an L staircase. Lowrie and Logan discuss how the frequency of encountering, and interacting with, information in visual/graphic format, including on the web, has increased our need for research on the role of spatial reasoning in students’ encoding and decoding of information in mathematics. To this end, they analyze the representational reasoning of students engaged in tasks that permit different types of representations, from diagrams to equations. Battista, Frazee, and Winer describe the spatial processes involved in reasoning about the geometric topics of measurement, shapes, and isometries. They introduce, and use in their analysis, the construct of *spatial-numerical linked structuring* as the coordinated process in which numerical operations on measurement numbers are linked to spatial structuring of, and operation on, the measured objects in a way that is consistent with properties of numbers and measurement.

As the chapters and commentaries illustrate, there are still fundamental differences between how researchers in psychology and mathematics education view and investigate the fundamental relations between spatial and mathematical reasoning. However, these differences provide fertile ground for exciting new investigations as each field respectively has advanced knowledge in some areas while leaving gaps in others. The commentaries are a starting point for identifying these points of contact and complementarity. We encourage readers to reflect on how the research in the two fields might be further integrated and how to build productive collaborations between the two sets of researchers. We thank all of our authors for taking a first step in this direction.

Michael T. Battista
Department of Teaching and Learning
The Ohio State University,
Columbus, OH, USA

Kelly S. Mix
Department of Human Development
and Quantitative Methodology
University of Maryland,
College Park, MD, USA

References

- Bruce, C., Davis, B., Sinclair, N., McGarvey, L., Hallowell, D., ..., Woolcott, G. (2017). Understanding gaps in research networks: using “spatial reasoning” as a window into the importance of networked educational research. *Educational Studies in Mathematics*, 95(2), 143–161.
- Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420–464). New York: Macmillan.
- Davis, B., & the Spatial Reasoning Study Group. (2015). *Spatial Reasoning in the Early Years: Principles, assertions, and speculations*. New York: Routledge.
- Geary, D. C., Hoard, M. K., Byrd-Craven, J., Nugent, L., & Numtee, C. (2007). Cognitive mechanisms underlying achievement deficits in children with mathematical learning disability. *Child Development*, 78(4), 1343–1359.
- Mix, K. S. (2010). Spatial tools for mathematical thought. In K.S. Mix, L. B. Smith, & M. Gasser (Eds.), *The spatial foundations of language and thought* (pp. 41–66). New York: Oxford University Press.
- Newcombe, N. S. (2010). Picture this: Increasing math and science learning by improving spatial thinking. *American Educator*, 34(2), 29–43.
- Wai, J., Lubinski, D., & Benbow, C. P. (2009). Spatial ability for STEM domains: Aligning over 50 years of cumulative psychological knowledge solidifies its importance. *Journal of Educational Psychology*, 101(4), 817–835.

Contents

Part I Psychological Perspectives

- 1 How Much as Compared to What: Relative Magnitude as a Key Idea in Mathematics Cognition** 3
Jamie Jirout and Nora S. Newcombe
- 2 From Intuitive Spatial Measurement to Understanding of Units.** 25
Eliza L. Congdon, Marina Vasilyeva, Kelly S. Mix,
and Susan C. Levine
- 3 Spatial Reasoning: A Critical Problem-Solving Tool in Children’s Mathematics Strategy Tool-Kit** 47
Beth M. Casey and Harriet Fell
- 4 More Space, Better Mathematics: Is Space a Powerful Tool or a Cornerstone for Understanding Arithmetic?** 77
Krzysztof Cipora, Philipp Alexander Schroeder, Mojtaba Soltanlou,
and Hans-Christoph Nuerk
- 5 What Processes Underlie the Relation Between Spatial Skill and Mathematics?** 117
Christopher Young, Susan C. Levine, and Kelly S. Mix

Part II Commentaries

- 6 Part I Commentary 1: Deepening the Analysis of Students’ Reasoning About Length** 151
Michael T. Battista, Leah M. Frazee, and Michael L. Winer
- 7 Part I Commentary 2: Visualization in School Mathematics Analyzed from Two Points of View** 165
A. Gutiérrez

8	Part I Commentary 3: Proposing a Pedagogical Framework for the Teaching and Learning of Spatial Skills: A Commentary on Three Chapters	171
	Tom Lowrie and Tracy Logan	
9	Part I Commentary 4: Turning to Temporality in Research on Spatial Reasoning	183
	Nathalie Sinclair	
Part III Educational Perspectives		
10	Analyzing the Relation Between Spatial and Geometric Reasoning for Elementary and Middle School Students	195
	Michael T. Battista, Leah M. Frazee, and Michael L. Winer	
11	Learning Through and from Drawing in Early Years Geometry	229
	Nathalie Sinclair, Joan Moss, Zachary Hawes, and Carol Stephenson	
12	The Interaction Between Spatial Reasoning Constructs and Mathematics Understandings in Elementary Classrooms	253
	Tom Lowrie and Tracy Logan	
13	Geometric Modeling of Mesospace Objects: A Task, its Didactical Variables, and the Mathematics at Stake	277
	Patricio Herbst and Nicolas Boileau	
14	Visualization Abilities and Complexity of Reasoning in Mathematically Gifted Students' Collaborative Solutions to a Visualization Task: A Networked Analysis	309
	A. Gutiérrez, R. Ramírez, C. Benedicto, M. J. Beltrán-Meneu, and A. Jaime	
Part IV Commentaries		
15	Part II Commentary 1: Mathematics Educators' Perspectives on Spatial Visualization and Mathematical Reasoning	341
	Beth M. Casey	
16	Part II Commentary 2: Disparities and Opportunities: Plotting a New Course for Research on Spatial Visualization and Mathematics	347
	Kelly S. Mix and Susan C. Levine	
17	Part II Commentary 3: Linking Spatial and Mathematical Thinking: The Search for Mechanism	355
	Nora S. Newcombe	

18 On the Multitude of Mathematics Skills: Spatial-Numerical Associations and Geometry Skill?	361
Krzysztof Cipora, Philipp A. Schroeder, and Hans-Christoph Nuerk	
Correction to: Visualizing Mathematics	E1
Index	371

Contributors

Michael T. Battista Department of Teaching and Learning, The Ohio State University, Columbus, OH, USA

M. J. Beltrán-Meneu Departamento de Educación, Jaume I University, Castellón, Spain

C. Benedicto Departamento de Didáctica de la Matemática, University of Valencia, Valencia, Spain

Nicolas Boileau Educational Studies Program, University of Michigan, Ann Arbor, MI, USA

Beth M. Casey Lynch School of Education, Boston College, Boston, MA, USA

Krzysztof Cipora Department of Psychology, University of Tuebingen, Tuebingen, Germany

LEAD Graduate School and Research Network, University of Tuebingen, Tuebingen, Germany

Eliza L. Congdon Department of Psychology, Bucknell University, Lewisburg, PA, USA

Harriet Fell College of Computer and Information Science, Northeastern University, Boston, MA, USA

Leah M. Frazee Department of Mathematical Sciences, Central Connecticut State University, New Britain, CT, USA

A. Gutiérrez Departamento de Didáctica de la Matemática, University of Valencia, Valencia, Spain

Zachary Hawes Ontario Institute for Studies in Education/University of Toronto, Toronto, ON, Canada

Patricio Herbst Educational Studies Program, University of Michigan, Ann Arbor, MI, USA

A. Jaime Departamento de Didáctica de la Matemática, University of Valencia, Valencia, Spain

Jamie Jirout Curry School of Education, University of Virginia, Charlottesville, VA, USA

Susan C. Levine Departments of Psychology, and Comparative Human Development and Committee on Education, University of Chicago, Chicago, IL, USA

Tracy Logan Faculty of Education, University of Canberra, Bruce, ACT, Australia

Tom Lowrie Faculty of Education, University of Canberra, Bruce, ACT, Australia

Kelly S. Mix Department of Human Development and Quantitative Methodology, University of Maryland, College Park, MD, USA

Joan Moss Ontario Institute for Studies in Education/University of Toronto, Toronto, ON, Canada

Nora S. Newcombe Department of Psychology, Temple University, Philadelphia, PA, USA

Hans-Christoph Nuerk Department of Psychology, University of Tuebingen, Tuebingen, Germany

Leibnitz-Institut für Wissensmedien, Tuebingen, Germany

LEAD Graduate School and Research Network, University of Tuebingen, Tuebingen, Germany

R. Ramírez Departamento de Didáctica de la Matemática, University of Granada, Granada, Spain

Philipp A. Schroeder Department of Psychology and Psychiatry, University of Tuebingen, Tuebingen, Germany

Department of Psychiatry and Psychotherapy, University of Tuebingen, Tuebingen, Germany

Nathalie Sinclair Faculty of Education, Simon Fraser University, Burnaby, BC, Canada

Mojtaba Soltanlou Department of Psychology, University of Tuebingen, Tuebingen, Germany

LEAD Graduate School and Research Network, University of Tuebingen, Tuebingen, Germany

Carol Stephenson Ontario Institute for Studies in Education/University of Toronto, Toronto, ON, Canada

Marina Vasilyeva Lynch School of Education, Boston College, Chestnut Hill, MA, USA

Michael L. Winer Department of Mathematics, Eastern Washington University,
Cheney, WA, USA

Christopher Young Consortium on School Research, University of Chicago,
Chicago, IL, USA