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Caroline Gruson · Vera Serganova

A Journey Through Representation Theory

From Finite Groups to Quivers via Algebras

Springer

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To Andrei Zelevinsky
who had such a great influence on our journey

Preface

Representation theory is a very active research topic in mathematics.

There are representations associated to several algebraic structures, representations of algebras and groups (finite or infinite). Roughly speaking, a representation is a vector space equipped with a linear action of the algebraic structure. For example, $\mathbb{C}^n$ is naturally a representation of the algebra of $n \times n$ matrices. A slightly more complicated example is the action of the group $GL(n, \mathbb{C})$ by conjugation on the space of $n \times n$-matrices.

When representations were first introduced, there was no tendency to classify all the representations of a given object. The first result in this direction is due to Frobenius, who was interested in the general theory of finite groups. If G is a finite group, a representation V of G is a complex vector space V together with a morphism of groups $\rho : G \to GL(V)$. One says V is irreducible if (1) there exists no non-zero proper subspace $W \subset V$ such that W is stable under all $\rho(g), g \in G$ and (2) $V \neq \{0\}$. Frobenius showed there are finitely many irreducible representations of G and that they are completely determined by their characters: the character of V is the complex-valued function $g \in G \mapsto \mathrm{Tr}(\rho(g))$ where Tr is the trace of the endomorphism. These characters form a basis of the complex-valued functions on G that are invariant under conjugation. Then Frobenius proceeded to compute the characters of symmetric groups in general. His results inspired Schur, who was able to relate them to the theory of complex finite dimensional representations of $GL(n, \mathbb{C})$ through the Schur–Weyl duality. In both cases, every finite dimensional representation of the group is a direct sum of irreducible representations (we say that the representations are completely reducible).

The representation theory of symmetric groups and the related combinatorics turn out to be very useful in a lot of questions. We decided to follow Zelevinsky and his book [38] and employ a Hopf algebra approach. This is an early example of categorification, which was born before the fashionable term categorification was invented.

Most of the results about representations of finite groups can be generalized to compact groups. In particular, once more, the complex finite dimensional representations of a compact group are completely reducible. Moreover, the regular representation in the space of continuous functions on the compact group contains every irreducible finite dimensional representation of the group. This theory was developed by H. Weyl and the original motivation came from quantum mechanics. The first examples of continuous compact groups are the group $SO(2)$ of rotations of the plane (the circle) and the group $SO(3)$ of rotations of the 3-dimensional space. In the former case, the problem of computing the Fourier series for a function on the circle is equivalent to the decomposition of the regular representation. More generally, the study of complex representations of compact groups helps to understand Fourier analysis on such groups.

If a topological group is not compact, for example, the group of real numbers under addition, the representation theory of such a group involves more complicated analysis (Fourier transform instead of Fourier series). The representation theory of real non-compact groups was initiated by Harish-Chandra and by the Russian school led by Gelfand. Here emphasis is on the classification of unitary representations due to applications from physics. It is also worth mentioning that this theory is closely related to harmonic analysis, and many special functions (such as Legendre polynomials) naturally appear in the context of representation theory.

In the theory of finite groups one can drop the assumption that the characteristic of the ground field is zero. This leads immediately to the loss of complete reducibility. This representation theory was initiated by Brauer, and it is more algebraic. If one turns to algebras, a representation of an algebra is, by definition, the same as a module over this algebra. Let k be a field. Let A be a k-algebra which is finite dimensional as a vector space. It is a well-known fact that A-modules are not, in general, completely reducible: for instance, if $A = k[X]/X^2$ and $M = A$, the module M contains kX as a submodule which has no A-stable complement. An indecomposable A-module is a non-zero module which has no non-trivial decomposition as a direct sum. It is also interesting to attempt a classification of A-modules. This is a very difficult task in general. Nevertheless, the irreducible A-modules are finite in number. The radical R of A is defined as the ideal of A which annihilates each of those irreducible modules. It is a nilpotent ideal. Assuming k is algebraically closed, the quotient ring A/R is a product of matrix algebras over k, $A/R = \Pi_i End_k(S_i)$ where S_i runs along the irreducible A-modules.

If G is a finite group, the algebra $k(G)$ of k-valued functions on G, the composition law being the convolution, is a finite dimensional k-algebra, with a zero radical as long as the characteristic of the field k does not divide the order of G. The irreducible modules of $k(G)$ are exactly the finite dimensional representations of the group G, and the action of G extends linearly to $k(G)$. This shows that all $k(G)$-modules are completely reducible (Maschke's theorem).

In order to study representations of finite dimensional k-algebras more generally, it is useful to introduce quivers. Let A be a finite dimensional k-algebra, denote $S_1, \ldots, S_n$ its irreducible representations, and draw the following graph, called the quiver associated to A: the vertices are labelled by the S_i's and we put l arrows between S_i and S_j, pointing at S_j, if $Ext^1(S_i, S_j)$ is of dimension l (the explicit definition of Ext^1 requires some homological algebra which is difficult to summarize in such a short introduction).

More generally, a quiver is an oriented graph with any number of vertices. Let Q be a quiver. A representation of Q is a set of vector spaces indexed by the vertices of Q together with linear maps associated to the arrows of Q. Those objects were first systematically used by Gabriel in the early 70s and studied by a lot of people ever since. The aim is to characterize the finitely represented algebras, or in other terms the algebras with a finite number of indecomposable modules (up to isomorphism).

When we get to representations of quivers (Chapters 7, 8 and 9), we will sometimes need some notions associated to algebraic groups. We do not provide a course in algebraic groups in this book; hence we refer the reader to the books of Humphreys and Springer cited in the bibliography.

Today, representation theory has many flavours. In addition to the above mentioned, one should add representations over non-Archimedean local fields with its applications to number theory, representations of infinite-dimensional Lie algebras with applications to number theory and physics, and representations of quantum groups. However, in all these theories certain main ideas appear again and again, very often in disguise. Due to technical details it may be difficult for a neophyte to recognize them. The goal of this book is to present some of these ideas in their most elementary incarnation.

We will assume that the reader is familiar with linear algebra (including the theory of Jordan forms and tensor products of vector spaces) and the basic theory of groups and rings.

The book is organized as follows. In the first two chapters we deal with the basic representation theory of finite groups over fields of characteristic zero. Some of these results extend to compact groups, see Chapter 3. Our aim in Chapter 4 is to provide examples where Fourier analysis plays a key role in unitary representations of locally compact groups. Since we need a lot of algebra later on, Chapter 5 is a collection of algebraic tools. Chapter 6 deepens the study of representations of symmetric groups and links them with representations of $GL_n(\mathbb{F}_q)$. Chapters 7 and 8 are an introduction to quivers and their representation theory. Finally, Chapter 9 gives some applications of quivers. Chapters 3 and 4 are not used in the rest of the book and can be omitted. We did not try to give a complete bibliography on the subject and cited only those books and papers which were directly used in the text.

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