A Visual Introduction to Differential Forms and Calculus on Manifolds

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To my parents, Daniel and Marlene Fortney, for all of their love and support.

Preface

Differential forms, while not quite ubiquitous in mathematics, are certainly common. And the role differential forms play appears in a wide range of mathematical fields and applications. Differential forms, and their integration on manifolds, are part of the foundational material with which it is necessary to be proficient in order to tackle a wide range of advanced topics in both mathematics and physics. Some upper-level undergraduate books and numerous graduate books contain a chapter on differential forms, but generally the intent of these chapters is to provide the computational tools necessary for the rest of the book, not to aid students in actually obtaining a clear understanding of differential forms themselves. Furthermore, differential forms often do not show up in the typically required undergraduate mathematics or physics curriculums, making it both unlikely and difficult for students to gain a deep and intuitive feeling for them. One of the two aims of this book is to address and remedy exactly this gap in the typical undergraduate mathematics and physics curriculums.

Additionally, it is during the second year and third year that undergraduate mathematics majors are making the transition from the concrete computation-based subjects generally found in high school and lower-level undergraduate courses to the abstract topics generally found in upper-level undergraduate and graduate courses. This is a tricky and challenging time for many undergraduate students, and it is during this period that most undergraduate programs see the highest attrition rates. Furthermore, while many undergraduate mathematics programs require mathematical structures or introduction to proofs class, there are also many programs do not. And often a single course meant to help students' transition from the concrete computations of calculus to the abstract notions of theoretical mathematics is not enough; a majority of students need more support in making this transition. The second aim of this book has been to help students make this transition to a mathematically more abstract and mature way of thinking.

Thus, the intended audience for this book is quite broad. From the perspective of the topics covered, this book would be completely appropriate for a modern geometry course; in particular, a course that is meant to help students make the jump from Euclidian/Hyperbolic/Elliptic geometry to differential geometry, or it could be used in the first semester of a two-semester sequence in differential geometry. It would also be appropriate as an advanced calculus course that is meant to help students' transition to calculus and analysis on manifolds. Additionally, it would be appropriate for an undergraduate physics program, particularly one with a more theoretical bent, or in a physics honors program; it could be used in a geometry for physics course. Finally, from this perspective, it is also a perfect reference for graduate students entering any field where a thorough knowledge of differential forms is necessary and who find they lack the necessary background. Though graduate students are not the intended audience, they could read and assimilate the ideas quite quickly, thereby enabling them to gain a fairly deep insight into the basic nature of differential forms before tackling more advanced material.

But from the perspective of helping undergraduate students make the transition to abstract mathematics, this book is absolutely appropriate for any and all second- or third-year undergraduate mathematics majors. Its mathematical prerequisites are light; a course in vector calculus is completely sufficient and the few necessary topics in linear algebra are covered in the introductory chapter. However, this book has been carefully written to provide undergraduates the scaffolding necessary to aid them in the transition to abstract mathematics. In fact, this material dove-tails with vector calculus, with which students are already familiar, making it a perfect setting to help students transition to advanced topics and abstract ways of thinking. Thus this book would be ideal in a second- or third-year course whose intent is to aid students in transitioning to upper-level mathematics courses.

As such, I have employed a number of different pedagogical approaches that are meant to complement each other and provide a gradual yet robust introduction to both differential forms in particular and abstract mathematics in general. First, I have made a great deal of effort to gradually build up to the basic ideas and concepts, so that definitions, when made, do not appear out of nowhere; I have spent more time exploring the "how" and "why" of things than is typical for most post-calculus math books. Additionally, the two major proofs that are done in this book (the generalized Stokes' theorem and the Poincaré lemma) are done very slowly and carefully, providing more detail than is usual. Second, this book tries to explain and help the reader develop, as much as possible, their geometric intuition as it relates to differential forms. To aid in this endeavor there are over 250 figures in the book. These images play a crucial role in aiding the student to understand and visualize the concepts being discussed and are an integral part of the exposition. Third, Students benefit from seeing the same idea presented and explained multiple times and from different perspectives; the repetition aids in learning and internalizing the idea. A number of the more important topics are discussed in both the $\mathbb{R}^n$ setting as well as in the setting of more abstract manifolds. Also, many topics are discussed from a visual/geometric approach as well as from a computational approach. Finally, there are over 200 exercises interspersed with the text and about 200 additional end-of-chapter exercises. The end of chapter questions are primarily computational, meant to help students gain familiarity and proficiency with the notation and concepts. Questions interspersed in the text range from trivial to challenging and are meant to help students genuinely engage with the readings, absorb fundamental ideas, and look carefully and critically at various steps of the computations done in the text. Taken together, these questions will not only allow students to gain a deeper understanding of the material, but also gain confidence in their abilities and internalize the essential notation and ideas.

Putting all of these pedagogical strategies together may result in an exposition that, to an expert, would seem at times to be unnecessarily long, but this book is based on my own experiences and reflections in both learning and teaching and is entirely written with students fairly new to mathematics in mind. I want my readers to truly understand and internalize these ideas, to gain a deeper and more accurate perception of mathematics, and to see the beautiful interconnectedness of the subject; I want my readers to walk away feeling that they have genuinely mastered a body of knowledge and not simply learned a set of disconnected facts.

Covering the full book is probably too much to ask of most students in a one-semester course, but there are a number of different pathways through the book based on the overall emphasis of the class. Provided below are the ones I consider most appropriate:

1. For schools on the quarter system or for a seminar class: 1 (optional), 2, 3, 4, 6, 7, 9 (optional).
2. Emphasizing differential forms and geometry: 1 (optional), 2–9, 10 (optional), 11, Appendix B1–2 (optional).
3. Emphasizing physics: 1 (optional), 2–7, 9, 11, 12, Appendix A (optional), Appendix B3–5 (optional).
4. Emphasizing the transition to abstract mathematics: 1 (optional), 2–4, 6–11.
5. Advanced students or as a first course to an upper-level sequence in differential geometry: 1 (optional), 2–11, Appendix A (optional), Appendix B (optional).

A word of warning, Appendix A on tensors was included in order to provide the proof of the global formula for exterior differentiation, a proof I felt was essential to provide in this book and which relies on the lie derivative. However, from a pedagogical perspective Appendix A is probably too terse and lacks the necessary examples to be used as a general introduction to tensors, at least if one wishes to cover anything beyond the mere definitions and basic identities. Instructors should keep this in mind when deciding whether or not to incorporate this appendix into their classes.

Finally, I would like to express my sincere appreciation to Ahmed Matar and Ron Noval for all of their invaluable comments and suggestions. Additionally, I would like to thank my friend Rene Hinojosa for his ongoing encouragement and support.

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