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# Spectral Action in Noncommutative Geometry

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*This book is dedicated to Alain Connes,*
*whose work has always been a fantastic*
*source of inspiration for us.*

# Preface

The Least Action Principle is among the most profound laws of physics. The action — a functional of the fields relevant to a given physical system — encodes the entire dynamics. Its strength stems from its universality: The principle applies equally well in every domain of modern physics including classical mechanics, general relativity and quantum field theory. Most notably, the action is a primary tool in model-building in particle physics and cosmology.

The discovery of the Least Action Principle impelled a paradigm shift in the methodology of physics. The postulates of a theory are now formulated at the level of the action, rather than the equations of motion themselves. Whereas the success of the 'New Method' cannot be overestimated, it raises a big question at a higher level: "Where does the action come from?" A quick look at the current theoretical efforts in cosmology and particle physics reveals an overwhelming multitude of models determined by the actions, which are postulated basing on different assumptions, beliefs, intuitions and prejudices. Clearly, it is the empirical evidence that should ultimately select the correct theory, but one cannot help the impression that our current models are only effective and an overarching principle remains concealed.

A proposal for such an encompassing postulate was formulated by Ali Chamseddine and Alain Connes in 1996 [5]. It reads [5, (1.8)]:

*The physical action should only depend upon the spectrum of* $\mathscr{D}$,

where $\mathscr{D}$ is a certain unbounded operator of geometrical origin. The incarnation of the *Spectral Action Principle* is very simple indeed:

$$S(\mathscr{D}, f, \Lambda) = \mathrm{Tr} f(|\mathscr{D}|/\Lambda),$$

with a given energy scale $\Lambda$ and a positive cut-off function $f$. Such a formulation provides a link with the current effective actions employed in field theoretic models and allows for a confrontation against the experimental data. The striking upshot of the spectral action is that, with a suitable choice of the operator $\mathscr{D}$, it allows one to retrieve the full Standard Model of particle physics in curved (Euclidean) spacetime [4, 6, 20]. This result attracted considerable interest in both physical and mathematical communities and triggered a far-reaching outflow of theoretical research. The most recent applications include Grand Unified Theories [7], modified Einstein gravity [15] and quantum gravity [10], to name only a few.

The formulation of the Least Action Principle dates back to the eighteenth century and the seminal works of Pierre de Maupertuis, Gottfried Leibniz and Leonhard Euler. The quest for its rigorous verbalisation sparked the development of the calculus of variations along with the Lagrangian and Hamiltonian formalisms. The modern formulation is expressed in the language of differential geometry.

The Spectral Action Principle is embedded in an even more advanced domain of modern mathematics — *noncommutative geometry*, pioneered and strongly pushed forward by Alain Connes [8, 9]. The idea that spaces may be quantised was first pondered by Werner Heisenberg in the 1930s (see [1] for a historical review) and the first concrete model of a 'quantum spacetime' was constructed by Hartland Snyder in 1949, extended by Chen-Ning Yang shortly afterwards. However, it took almost half a century for the concepts to mature and acquire the shape of a concrete mathematical structure. By now, noncommutative geometry is a well-established part of mathematics.

Noncommutative geometry à la Connes sinks its roots not only in the Riemannian geometry, but also in the abstract framework of operator algebras. Its conceptual content is strongly motivated by two fundamental pillars of physics: general relativity and quantum mechanics, explaining why it has attracted both mathematicians and theoretical physicists. It offers a splendid opportunity to conceive 'quantum spacetimes' turning the old Heisenberg's dream into a full-bodied concept.

In this paradigm, geometry is described by a triplet $(\mathscr{A}, \mathscr{H}, \mathscr{D})$, where $\mathscr{A}$ is a not necessarily commutative algebra, $\mathscr{D}$ is an operator (mimicking the Dirac operator on a spin manifold) both acting on a common scene — a Hilbert space $\mathscr{H}$. Thus, by essence, this geometry is spectral. The data of a spectral triple $(\mathscr{A}, \mathscr{H}, \mathscr{D})$ covers a huge variety of different geometries. The classical (i.e. commutative) case includes primarily the Riemannian manifolds, possibly tainted by boundaries or singularities, but also discrete spaces, fractals and non-Hausdorff spaces and when $\mathscr{A}$ is noncommutative, the resulting 'pointless' geometries, with the examples furnished by the duals of discrete groups, dynamical systems or quantum groups to mention but a few.

At this point, one should admit that the simple form of the spectral action is deceiving — an explicit computation would require the knowledge of the full spectrum of the operator $\mathscr{D}$, which is hardly ever the case. Nevertheless, one can extract a great deal of physically relevant information by studying the asymptotics of $S(\mathscr{D}, f, \Lambda)$ when $\Lambda$ tends to infinity. The key tool to that end is the renowned heat kernel method fruitfully employed in classical and quantum field theory, adapted here to the noncommutative setting. Beyond the (almost) commutative case, the latter is still a vastly uncharted water. It is our primary intent to provide a faithful map of the mathematical aspects behind the spectral action. Whereas the physical motivation will be present in the backstage, we leave the potential applications to the Reader's invention. To facilitate the latter, we recommend to have a glimpse into the textbooks [15, 20] and references therein.

The plan of our guided tour presents itself as follows:

In the first chapter, the basics of noncommutative geometry à la Connes are laid out. Chapter 2 is designed to serve as a toolkit with several indispensable notions related to spectral functions and their functional transforms. Therein, the delicate notion of an asymptotic expansion is carefully detailed, both in the context of functions and distributions. With Chap. 3, we enter into the hard part of this book, which unveils the subtle links between the existence of asymptotic expansions of traces of heat operators and meromorphic extensions of the associated spectral zeta functions. While trying to stay as general as possible, we illustrate the concepts with friendly examples. Therein, the large energies' asymptotic expansion of the spectral action is presented in full glory. Chapter 4 is dedicated to the important concept of a fluctuation of the operator $\mathscr{D}$ by a 'gauge potential' and its impact on the action. In terms of physics, this means a passage from 'pure gravity' to a full theory vested with the all admissible gauge fields. In terms of mathematics, it involves rather advanced manipulations within the setting of abstract pseudodifferential operators, which we unravel step by step. We conclude in Chap. 5 with a list of open problems, which — in our personal opinion — constitute the main stumbling blocks in the quest of understanding the mathematics and physics of the Spectral Action Principle. We hope that these would inspire the Reader to have his own take on the subject. The bulk of the book is complemented with a two-part Appendix. Section A contains further auxiliary tools from the theory of pseudodifferential operators, including a detailed derivation of the celebrated heat kernel expansion. In Section B, we present examples of spectral geometries of increasing complexity: spheres, tori, noncommutative tori and a quantum sphere.

This book is devoted to the spectral action, which is only a small offspring in the vast domain of noncommutative geometry. Therefore, when introducing the rudiments of Connes' theory, we are bound to be brief and focus on the specific aspects related to the spectral action. We refer the Reader to the textbooks for a complete introduction on noncommutative geometry [9, 11, 13, 14, 21].

Let us also warn the Reader that, although we have designed the book to be as self-contained as possible, some mathematical prerequisites are indispensable to grasp the presented advanced concepts. The Reader should be acquainted with the basics of functional analysis, including, in particular, the spectral theory of unbounded operators on Hilbert spaces (e.g. [2, 17, 18]) and the rudiments of operator algebras (e.g. [3, 12]). Some intuitions from global differential geometry (e.g. [16]) and the theory of pseudodifferential operators (e.g. [19]) may also prove useful.

Our ultimate purpose is not only to provide a rigid first course in the spectral action, but also to charm the Reader with the marvellous interaction between mathematics and physics encapsulated in this apparently simple notion of spectral action. Let *res ipsa loquitur…*

During the years spent in the realm of noncommutative geometry, we have collaborated with a number of our close colleagues: Driss Essouabri, Nicolas Franco, Victor Gayral, José Gracia-Bondía, Michael Heller, Cyril Levy, Thierry Masson, Tomasz Miller, Andrzej Sitarz, Jo Varilly, Dmitri Vassilevich, Raimar Wulkenhaar, Artur Zając. We also took benefits from discussions with Alain Connes. Moreover, Tomasz was a scrupulous proofreader and Thierry was a great help with the LaTeX typesetting. It is our pleasure to cordially thank all of them, as without their kind support this book could not come into being.

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