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B-Model Gromov-Witten Theory

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Preface

This book is the product of a special semester on B-model Gromov-Witten theory held at the University of Michigan in winter 2014. The goal of the semester was to bring together experts (including both mathematicians and physicists) on various aspects of mirror symmetry in order to better appreciate how their different perspectives fit together into a coherent—yet still not entirely well-understood—story.

Mirror symmetry, an equivalence between two versions of string theory, emerged in the 1980s as a duality in theoretical physics. From a physical perspective, the duality between the A-model of a manifold X and the B-model of its mirror dual $X^\vee$ was natural to expect because the two theories encode the same physics. Mathematically, though, it has taken years to specify the precise data that the A- and B-models should capture, and the correspondences that have resulted have been striking and unexpected. The A-model associated with X, for example, can be viewed mathematically as encoding the enumerative geometry of curves inside of X, leading to the advent of Gromov-Witten theory. When one restricts to genus-zero curves, the B-model can be understood in terms of period integrals on $X^\vee$, which have been classically studied and often explicitly computed. In this way, genus-zero mirror symmetry has led (as in the celebrated example of the quintic threefold) to beautifully explicit answers to some of enumerative geometry's long-standing questions. These answers were mathematically only conjectural when they were first proposed (by physicists Candelas et al. [2]), but a mathematical proof of genus-zero mirror symmetry followed in the 1990s from the work of Givental [4, 5] and Lian–Liu–Yau [12]. This was a major event, leading to the birth of mirror symmetry as a mathematical subject.

Motivated by this success, one would hope to develop an analogous correspondence in all genera, but the pursuit of higher-genus mirror symmetry has been a very difficult task. On the A-side, while Gromov-Witten theory has a firm mathematical foundation in all genera, its computation in genus beyond zero remains one of the hardest problems in geometry and physics. On the B-side, even the theoretical foundations of the higher-genus theory are mathematically not fully understood. One can attempt to forge ahead, nevertheless, using physical methods; indeed, as early as 1993, Bershadsky et al. [1] formulated a higher-genus B-model theory (now known as BCOV theory) in physical language and proposed many of its key properties, such as the famous "holomorphic anomaly equation." Furthermore, they calculated the B-model generating function explicitly in genera one and two, which implied, by way of mirror symmetry, a conjectural closed formula for the generating functions F_g of genus-g Gromov-Witten invariants of the quintic threefold when $g \in \{1, 2\}$. It took ten years before a mathematical proof of the BCOV formula for F_1 was given by Zinger [15], and another ten years for the analogue in genus two, by Guo–Janda–Ruan [8].

Meanwhile, physicists have continued their B-model calculations with great success. A B-model formula for F_3 was calculated by Katz–Klemm–Vafa in 1999 [10], and a number of structural results on the generating functions F_g were predicted on physical grounds. For example, a fundamental physical result of Yamaguchi–Yau [14] states that F_g is a polynomial in five generators, constructed explicitly from the period integrals of the mirror, of which four are holomorphic limits of certain non-holomorphic objects in the B-model; the holomorphic anomaly equation of BCOV theory can be recast into equations relating these four generators. Using the holomorphic anomaly equation, together with other physical predictions regarding the structure of F_g (the conifold gap condition, for example, and orbifold regularity), Huang et al. [9] pushed the physical calculation of F_g to all $g \leq 51$ for the quintic threefold and to other large bounds for targets such as "one-parameter models" and elliptically fibered Calabi–Yau threefolds.

Compared to this stunning success in physics, mathematical progress has been frustratingly slow. One of the reasons is that mathematical understanding of the BCOV B-model theory in higher genus, including the non-holomorphicity of the B-model generating function, remains limited. We hope, in this book, to help change this state of affairs by collecting some of what is known and providing a reference for future study.

The organization of the book is as follows. Chapter "Mirror Symmetry Constructions" (contributed by Emily Clader and Yongbin Ruan) outlines the various ways in which mirror pairs have been constructed. These include the Batyrev construction, which produces a mirror pair of Calabi–Yau hypersurfaces in toric varieties; the Hori–Vafa construction, in which the mirror to a semi-Fano complete intersection in a toric variety is produced as a "Landau–Ginzburg model" (a variety X together with a polynomial function $X \to \mathbb{C}$ known as the superpotential); and the Berglund–Hübsch–Krawitz construction, which produces a mirror pair of Landau–Ginzburg models. For each of these constructions, mirror symmetry is discussed in its most basic manifestation: the "state space correspondence," an isomorphism between bi-graded vector spaces associated with each theory. For the geometric theory, the state space is simply the (orbifold) cohomology of the hypersurface or complete intersection, with different bi-gradings on the A- and B-sides, while in the Landau–Ginzburg theory, it is a certain orbifold Jacobian ring of the superpotential. Mirror symmetry provides a grading-preserving isomorphism between the A- and B-model state spaces. The upshot, for example, in Batryev mirror symmetry is that the Hodge diamonds of a mirror pair of hypersurfaces are related by rotation, which, in the case of Calabi–Yau threefolds, reduces to the statement that mirror symmetry exchanges the two Hodge numbers $h_{1,1}$ and $h_{2,1}$. These are the dimensions, respectively, of the space of infinitesimal deformations of the Kähler structure and the space of infinitesimal deformations of the complex structure, providing a first hint that the state space correspondence is a shadow of a deeper duality.

The bulk of the book is chapter "The B-model Approach to Topological String Theory on Calabi-Yau n-Folds" (contributed by Albrecht Klemm). We focus on a particular mirror symmetry construction—the Batyrev construction—but upgrade our perspective from the state space correspondence to the much richer data of the topological A- and B-models, which are constructed physically out of two different topological twists of an $N = 2$ two-dimensional supersymmetric quantum field theory. In particular, a major part of this long chapter is devoted to providing a full (physical) account of BCOV B-model theory and its key properties.

According to the physical technique of supersymmetric localization, the A-model path integral of a compact Calabi–Yau threefold X localizes to the space of holomorphic curves of prescribed area, so it depends on the complexified Kähler structure (not on the complex structure) of X. The resulting mathematical theory is Gromov-Witten theory. For the B-model, the path integral localizes to the space of constant maps, so the complication lies in the contribution of the transverse directions; in particular, it depends only on the complex structure (not on the Kähler structure) of the target. Thus, for a threefold X with mirror $X^\vee$, the goal of mirror symmetry may be viewed as the construction, at least locally,[1] of a function

$$t_* : \mathcal{M}_{\mathrm{cs}}(X^\vee) \to \mathcal{M}_{\mathrm{cks}}(X)$$

from the moduli space of complex structures on $X^\vee$ to the moduli space of complexified Kähler structures on X, such that t_* relates the A- and B-model amplitudes. Given such a function, the classical duality interchanging $h_{1,1}$ and $h_{2,1}$ (as discussed in chapter "Mirror Symmetry Constructions") can be recovered by taking tangent spaces on both sides.

More specifically, the generating function of Gromov-Witten invariants can be viewed (modulo certain convergence issues) as a holomorphic function defined within the "large radius" region $V \subseteq \mathcal{M}_{\mathrm{cks}}(X)$, and one goal of BCOV B-model theory is to define a similar function on $\mathcal{M}_{\mathrm{cs}}(X^\vee)$ and to prove that these two functions agree under the mirror map t_*. Locally, the study of the complex structure of $X^\vee$ is equivalent to its variation of Hodge structure. For genus-zero worldsheets, this reduces to studying the dependence of period integrals on complex structure, a classical subject in algebraic geometry that can be handled explicitly via the Picard–Fuchs differential equations and monodromy techniques. In higher genus, the entire

[1] Indeed, this correspondence can only be local, because the moduli space of complexified Kähler structures is a ball centered around a point known as the "large radius limit", whereas the moduli space of complex structures carries nontrivial topology. The effort to extend the moduli space of complexified Kähler structures leads to other subjects, such as the gauged linear sigma model, which are outside the scope of this book.

BCOV theory comes into play, and it is much less well known to the mathematical community. Much of chapter "The B-model Approach to Topological String Theory on Calabi-Yau n-Folds" is devoted to explaining these ideas.

In the construction of the B-model generating function, a number of key differences between the A- and B-model theories become apparent. First, the A-model generating function is always holomorphic, whereas the B-model generating function is not; its non-holomorphic dependence is the subject of the holomorphic anomaly equation. Second, in light of the nontrivial topology on the moduli space of complex structures, the B-model generating function can be viewed as a global object—more precisely, a section of a certain line bundle. This naturally leads to connections between the B-model theory and the theory of modular forms, which are holomorphic sections of the same bundle. These two aspects of the theory are intimately related; in dimension one, for example, the holomorphic anomaly equation implies that the B-model generating function is a quasi-modular form. On Calabi–Yau threefolds, moreover, the holomorphic anomaly equation is a powerful computational tool, leading to such structural features as Yamaguchi–Yau's prediction that F_g is a polynomial in certain canonical generators constructed from period integrals. One of the key issues discussed in chapter "The B-model Approach to Topological String Theory on Calabi-Yau n-Folds" is the comparison between the A- and B-model generating function, which involves the delicate procedure of taking a "holomorphic limit" of the B-model generating function using the geometry of the moduli space of complex structures.

The account of BCOV theory provided in chapter "The B-model Approach to Topological String Theory on Calabi-Yau n-Folds" is physical in nature, whereas a mathematical construction of the theory—though still far from complete—has been initiated by Costello and Li [3, 11], whose work we encourage interested readers to consult. A fundamental feature of both the physical and mathematical constructions is that the higher-genus theory is defined as a quantization from genus zero. In order to explain this in mathematical terms, we digress slightly in chapter "Geometric Quantization with Applications to Gromov-Witten Theory" (contributed by Emily Clader, Nathan Priddis, and Mark Shoemaker) to explain the topic of geometric quantization. This is a procedure for producing a Hilbert space $\mathbb{H}_V$ from a polarized symplectic vector space V (finite- or infinite-dimensional), which is functorial in the sense that a symplectic linear transformation $T : V \to W$ gives rise to an operator $\widehat{T} : \mathbb{H}_V \to \mathbb{H}_W$. The physical meaning of this procedure, and the explanation for its name, comes from the passage from a classical theory, in which V represents the space of states, to the associated quantum theory with state space $\mathbb{H}_V$.

Quantization also has deep mathematical significance, encoding the relationship between the genus-zero Gromov-Witten theory of certain targets X and their higher-genus theory. In this case, one sets $V = V_X = H^*(X; \Lambda)((z^{-1}))$, where Λ is a Novikov ring and z a formal parameter. The genus-zero Gromov-Witten invariants of X can be packaged into a Lagrangian submanifold $\mathcal{L}_X \subseteq V_X$, and the higher-genus invariants into a total descendant potential $\mathcal{D}_X \in \mathbb{H}_{V_X}$. If the targets satisfy the condition of "semisimplicity", then a symplectic transformation $T : V_X \to V_Y$ for which $T(\mathcal{L}_X) = \mathcal{L}_Y$ has $\widehat{T}(\mathcal{D}_X) = \mathcal{D}_Y$. In particular, if one can produce a symplectic transformation taking the Gromov-Witten theory of X to the theory of a finite collection of points, then the all genera Gromov-Witten theory of X can be deduced via quantization from that simplest of targets. These ideas, which were developed in the deep foundational work of Givental [6, 7] and Teleman [13], are discussed in the second half of chapter "Geometric Quantization with Applications to Gromov-Witten Theory".

Equipped with the quantization machinery, in chapter "Some Classical/Quantum Aspects of Calabi-Yau Moduli" (contributed by Si Li) we turn to a mathematical perspective on the higher-genus B-model developed in [3]. The main idea is that one should work on the chain level rather than in cohomology. More precisely, the moduli space $\mathcal{M}_{\text{cs}}(X)$ of complex structures can be described mathematically in terms of the chain complex of "polyvector fields" on X. From here, via Kyoji Saito's theory of primitive forms, one obtains a Frobenius manifold structure on $\mathcal{M}_{\text{cs}}(X)$ and, in particular, a potential function $\mathcal{F}$. Within the "large complex structure" region $U \subseteq \mathcal{M}_{\text{cs}}(X)$, this potential function has been identified in a large class of examples with the generating function of genus-zero Gromov-Witten invariants on the mirror $X^\vee$. To obtain the higher-genus B-model, then, one applies the quantization procedure described in chapter "Geometric Quantization with Applications to Gromov-Witten Theory". This is simply a definition, but at the end of chapter "Some Classical/Quantum Aspects of Calabi-Yau Moduli", we discuss the case where X is an elliptic curve, in which one can prove that the higher-genus B-model correlation functions produced via quantization indeed agree in the large complex structure limit with the higher-genus Gromov-Witten invariants of X.

An alternative mathematical development of the higher-genus B-model is presented in chapter "Eynard-Orantin B-model and Its Application in Mirror Symmetry" (contributed by Bohan Fang), by way of the Eynard–Orantin topological recursion. Specifically, one defines a "spectral curve" as an affine algebraic curve $\Sigma \subseteq (\mathbb{C}^*)^2$ equipped with a certain extra structure, and the Eynard–Orantin formalism recursively defines a collection of symmetric meromorphic differential forms $\omega_{g,n}$ on the n-fold product $\overline{\Sigma}^n$ of a compactification $\overline{\Sigma}$. (In fact, formal spectral curves can be defined in a neighborhood of each semisimple point of a generically semisimple Frobenius manifold, and it has been shown that the Eynard–Orantin recursion is equivalent in this case to Givental's quantization.) For certain target spaces X, an associated spectral curve Σ can be defined, and the Eynard–Orantin invariants of Σ can be viewed as a higher-genus B-model of Σ. In particular, the mirror symmetry prediction that the A-model of X and the B-model of Σ agree has been verified in a number of cases, including all toric Calabi–Yau threefolds. One application of this view on the B-model is that it confirms the modularity properties predicted by physicists, as these properties hold in the Eynard–Orantin theory.

Chapter "The Total Ancestor Potential in Singularity Theory" (contributed by Todor Milanov) addresses the question of when the B-model potential function, defined via quantization as in chapter "Some Classical/Quantum Aspects of Calabi-Yau Moduli", is analytic. This is a particularly desirable property that is relevant, for example, in the Landau–Ginzburg/Calabi–Yau correspondence, which relates the

A-model in two different regions of $\mathcal{M}_{\text{cks}}$ (the large radius region, corresponding to Gromov-Witten theory, and the orbifold region, corresponding to the Fan–Jarvis–Ruan–Witten theory of an associated singularity) by passing through a mirror symmetric comparison of both theories to a global B-model. Specifically, chapter "The Total Ancestor Potential in Singularity Theory" focuses on Frobenius manifolds arising via the universal unfolding of a polynomial function $f : \mathbb{C}^{n+1} \to \mathbb{C}$, in which the Frobenius structure corresponds to the choice of a primitive form; this is the analogue in the orbifold region of the Frobenius manifold of chapter "Some Classical/Quantum Aspects of Calabi-Yau Moduli". In contrast to the Calabi–Yau setting considered in chapter "The B-model Approach to Topological String Theory on Calabi-Yau n-Folds", this Frobenius manifold is generically semisimple, which makes the quantization operation much better behaved. In particular, the B-model generating function is always analytic at semisimple points, so the question is whether it extends analytically across the non-semisimple locus. For the Frobenius manifolds discussed in chapter "The Total Ancestor Potential in Singularity Theory", this is indeed the case. Furthermore, when f is quasi-homogeneous and satisfies a condition known as "invertibility," the ancestor potential of the above Frobenius manifold can be identified with the generating function of all genera Fan–Jarvis–Ruan–Witten invariants of f.

Finally, in chapter "Lecture Notes on Bihamiltonian Structures and Their Central Invariants" (contributed by Si-Qi Liu), we turn to a rather different aspect of the B-model: its connection to integrable systems. In the early days of Gromov-Witten theory, Witten conjectured that the generating function of certain intersection numbers on the moduli space of curves is a tau-function of the KdV hierarchy. This conjecture was soon proven by Kontsevich, work for which he was awarded a Fields Medal and that generated a great deal of interest in the interplay between Gromov-Witten theory and integrable systems. A prominent approach to this subject, due to Dubrovin and Zhang, is to develop an axiomatization of the integrable hierarchies arising in geometry. Through this technique, Dubrovin and Zhang proved that the higher-genus generating function associated which a semi-simple Frobenius manifold is uniquely determined by the condition that it is a deformation of the genus-zero generating function and it satisfies a system of differential equations known as the Virasoro constraints. This observation directly motivated Givental's work on quantization. Dubrovin and Zhang pushed their theory much further, however, classifying all possible integrable systems for the higher-genus theory up to "gauge transformations." Since then, their technique has become a standard method for studying the integrable systems associated with semisimple Frobenius manifolds. Chapter "Lecture Notes on Bihamiltonian Structures and Their Central Invariants" gives an account of this fascinating story.

The material presented in this book is by no means complete. For example, it only scratches the surface of Costello–Li's B-model and does not cover any aspects of homological mirror symmetry or the derived category. The reader should take this book as a starting point rather than a definitive reference. As the editors, we wish to thank all the people who contributed to the book, without whose efforts it would not have been possible.

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