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Convex Duality and Financial Mathematics

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To Carol and Olivia
To Lilly and Charles.
And in memory of Jonathan Borwein
(1951–2016) with respect.

Preface

Convex duality plays an essential role in many important financial problems. For example, it arises both in the minimization of convex risk measures and in the maximization of concave utility functions. Together with generalized convex duality, they also appear when an optimization is not immediately apparent, for instance in implementing dynamic hedging of contingent claims. Recognizing the role of convex duality in financial problems is crucial for several reasons. First, considering the primal and dual problem together gives the financial modeler the option to tackle the more accessible problem first. Usually, knowledge of the solution of one helps in solving the other. Moreover, the solution to the dual problem can usually be given a financial interpretation. As a result, the dual problem often illuminates an alternative perspective, which is not easily achieved by examining the primal problem in isolation. When flipping from the primal to the dual, a surprise insight typically awaits, irrespective of past experience. Finally, as an added benefit, the primal and the dual can often be paired together to provide better numerical solutions than when either side is considered in isolation.

The goal of this book is to provide a concise introduction to this growing research field. Our target audience is graduate students and researchers in related areas. We begin in Chapter 1 with a quick introduction of convex duality and related tools. We emphasize the relationship between convex duality and the Lagrange multiplier rule for constrained optimization problems. We then give a quick overview of the intrinsic duality relationship in several diverse financial problems.

In Chapter 2, we consider the simplest possible financial market model. In particular, we consider a one-period economy with a finite number of possible states. Using this simple financial market model, we showcase convex duality in a number of important financial problems. We begin with the Markowitz portfolio theory, which involves a particularly simple convex programming problem: optimizing a quadratic function with linear constraints. Duality plays two important roles in Markowitz portfolio theory. First, while the primal problem may involve hundreds or even thousands of variables representing the risky assets potentially included in the portfolio, the dual problem has only two variables related to the two constraints on the initial endowment and the expected return. In fact, the key observation of Markowitz is that one can evaluate the performance of a portfolio in the dual space using the variance-expected return pair. Second, the duality relationship between the primal Markowitz portfolio problem and its dual helps us to understand that the set of optimal portfolios is an affine set, which leads to the important two-fund theorem. The core methodology of optimizing a quadratic function with linear constraints was also used in the capital asset pricing model, which leads to the widely used Sharpe ratio. Duality also plays a crucial role in this problem.

Next, we consider portfolio optimization from the perspective of maximizing expected utility. There has been a very long history of using utility functions in economics. In financial problems, utility functions are increasing concave functions of wealth. The concavity of the utility function captures the risk aversion of an investor. Arrow and Pratt introduced widely used measures of the level of risk aversion. It turns out that there is a precise way of using generalized convexity to characterize Pratt–Arrow risk aversion. This application illustrates the relevance of generalized convexity in dealing with financial problems. It is even more interesting to consider the dual of the expected utility maximization problem. It turns out that in the absence of arbitrage, solutions to the dual problem are in essence the equivalent martingale measures (also called risk-neutral probabilities), which are widely used in pricing financial derivatives. Considering the expected utility maximization problem along with its dual leads us to rediscover the fundamental theorem of asset pricing. An added benefit of this alternative approach is that martingale measures can be related to the risk aversion of agents in the market.

The last application that we cover in Chapter 2 concerns the dual representation of coherent risk measures. Coherent risk measures are motivated by the common regulatory practice of assigning each position in a risky asset with the appropriate amount of cash reserves. Hence, they are widely used to analyze risks. Mathematically, a coherent risk measure is characterized by a sublinear function: a convex function with positive homogeneity. It is well known that the dual of a sublinear function is an indicator function. Thus, using dual representation, a coherent risk measure is just the support function of a closed convex set. Financially, we can view the generating set of a coherent risk measure as the probabilities assigned to risky scenarios in a stress test. Duality also generates numerical methods for calculating some important coherent risk measures such as the conditional value at risk.

We expand our discussion to a more general multiperiod financial market model in Chapter 3. This more general setting allows us to model dynamic trading. The added complexity in dealing with a multiperiod model mainly involves capturing the increase in information using an information structure. After laying out the multiperiod financial market model, we show that the fundamental theorem of asset pricing also arises in a multiperiod financial market model. After that we also discuss two new topics: super (sub) hedging and conic finance. In general, the absence of arbitrage leads to multiple (usually infinitely many) pricing martingale measures in an incomplete market. Thus, the no arbitrage principle usually determines a price range for a contingent claim with upper and lower bounds, which are given by the supremum and the infimum of the expectation of the payoff under the martingale measures, respectively. If a market price falls outside of these bounds, then an arbitrage opportunity occurs. It turns out that the dual solution to the optimization problem of finding the upper or lower no arbitrage bounds provides a trading strategy that one can use to take advantage of such an arbitrage opportunity. Conic finance is used to describe financial markets for which the absolute value of the price depends on whether one is buying or selling. In other words, conic finance describes realistic financial markets with a strictly positive bid-ask spread. In such a model, the cash flows that can be achieved from implementing acceptable trading strategies form a convex cone. This observation provides the rationale for the name conic finance. Despite the added complication of dealing with a conic constraint, we show that most of the duality relationships that are observed under zero bid–ask spread still prevail when the spread is positive.

We then move to continuous-time financial models in Chapter 4. The most noteworthy duality relationship developed in this chapter is the observation that the classical Black-Scholes formula for pricing a contingent claim with a convex payoff is, in fact, a Fenchel-Legendre transform. We show that the function describing cash borrowings while delta hedging a short position in a contingent claim is just the Fenchel conjugate of the contingent claim pricing function. The flip side is that the contingent claim pricing function can itself be viewed as a Fenchel conjugate of the function describing these cash borrowings. This provides a new perspective on the convex function linking the price of the contingent claim to the underlying spot price. With the availability of many tradable contingent claims such as those embedded in ETFs, the ability to dynamically hedge a contingent claim with other contingent claims is increasingly becoming a financial reality. Interestingly, when using contingent claims as hedging instruments, one discovers a similar duality relationship between the contingent claim pricing function and the cash borrowings function in terms of generalized convexity. Many useful applications are also discussed in this chapter. We examine the convexity and generalized convexity of the Bachelier and Black-Scholes option pricing formulae with respect to volatility as well. Generalizations of these properties might be useful in dealing with financial products related to volatility and be a potentially fruitful future research direction.

The material in this book grew out of slides used to teach a joint doctoral seminar at New York University's Courant Institute in the fall of 2015. Part of the materials has also been used previously for graduate topic courses on optimization and modeling at Western Michigan University. We thank our colleagues at both NYU and WMU for providing us with supportive research environments. Professor Robert Kohn helped to arrange us becoming neighbors, which facilitated our collaboration in no small part. Conversations with Professors Marco Avellaneda, Jonathan Goodman, and Fang-Hua Lin have been most helpful. We are also indebted to the participants of these courses for many stimulating discussions. In particular, we thank Monty Essid, Tom Li, Matthew Foreman, Sanjay Karanth, Jay Treiman, Mehdi Vazifadan, and Guolin Yu whose detailed comments on various parts of our lecture notes have been incorporated into the text.

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