SpringerBriefs in Mathematics

SpringerBriefs in Mathematics showcases expositions in all areas of mathematics and applied mathematics. Manuscripts presenting new results or a single new result in a classical field, new field, or an emerging topic, applications, or bridges between new results and already published works, are encouraged. The series is intended for mathematicians and applied mathematicians.

More information about this series at http://www.springer.com/series/10030

Robert A. McCoy · Subiman Kundu
Varun Jindal

Function Spaces with Uniform, Fine and Graph Topologies

Springer

Robert A. McCoy
Department of Mathematics
Virginia Polytechnic Institute and State University
Blacksburg, VA
USA

Varun Jindal
Department of Mathematics
Malaviya National Institute of Technology
Jaipur, Rajasthan
India

Subiman Kundu
Department of Mathematics
Indian Institute of Technology Delhi
New Delhi, Delhi
India

ISSN 2191-8198 ISSN 2191-8201 (electronic)
SpringerBriefs in Mathematics
ISBN 978-3-319-77053-6 ISBN 978-3-319-77054-3 (eBook)
https://doi.org/10.1007/978-3-319-77054-3

Library of Congress Control Number: 2018935873

Mathematics Subject Classification (2010): 54C35, 54E05, 54D65, 54E15, 54D05, 54E50

Printed on acid-free paper

This Springer imprint is published by the registered company Springer International Publishing AG part of Springer Nature
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

To My Wife Carolyn

Robert A. McCoy

**

To the Memory of My Parents Krishnagopal Kundu and Mira Kundu

Subiman Kundu

**

To the Memory of My Grandfather Roshan Lal Gupta

Varun Jindal

Contents

Symbols

$\in$	Belongs to
$\subseteq$	Subset or equal
$\cup$, $\cap$	Union, intersection
$X\backslash E$	The complement of E in X
$\mathbb{N}$	The set of natural numbers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}$	The real line
$\mathbb{R}^n$	The n-dimensional Euclidean space
$\mathbb{I}$	The closed interval $[-1, 1]$
$\square$	End of a proof
$\oplus$	Topological sum
$\|\| \cdot \|\|$	Norm
A^B	If A and B are two non-empty sets, then A^B denotes the set of all functions from B to A
$(x_n)_{n\in\mathbb{N}}$ or $\{x_n : n \in \mathbb{N}\}$	A sequence in a non-empty set, occasionally it may be denoted by (x_n)
ω	The first infinite ordinal
ω_1	The first uncountable ordinal
$f\|_A$	The restriction of f to A where $f : X \to Y$ is a function,

For the following notations, X is a topological space.

$\overline{A}$ or $cl_X A$	The closure of A in X
$intA$ or A°	The interior of A in X
$B(x, \varepsilon)$	The open ball centered at x with radius $\varepsilon > 0$
$\overline{B}(x, \varepsilon)$	The closed ball centered at x with radius $\varepsilon > 0$
βX	The Stone–Čech compactification of X, where X is a Tychonoff space
G_δ-set	A countable intersection of open sets in X

$\chi(X)$	Character of X
$d(X)$	Density of X
$c(X)$	Cellularity of X
$w(X)$	Weight of X
$L(X)$	Lindelöf number of X
$e(X)$	Extent of X
$C(X, Y)$	The set of all continuous functions from a Tychonoff space X to a space Y
$H(X)$	The set of all self-homeomorphisms on a metric space X
$C(X)$	The set of all real-valued continuous functions on a Tychonoff space X
$C_+(X)$	The set of all positive continuous functions defined on X
$LC_+(X)$	The set of all positive lower semi-continuous functions defined on X
$C^*(X)$	The set of all bounded functions in $C(X)$
$H^+(\mathbb{R})$	The set of all increasing homeomorphisms from $\mathbb{R}$ to $\mathbb{R}$
$H^-(\mathbb{R})$	The set of all decreasing homeomorphisms from $\mathbb{R}$ to $\mathbb{R}$
$\mathscr{F}(X)$	The set of all finite subsets of X
$\mathscr{K}(X)$	The set of all compact subsets of X
0_X	The constant zero function in $C(X, Y)$, where Y is a normed linear space
$C_p(X, Y)$	The space $C(X, Y)$ equipped with the point-open topology
$C_k(X, Y)$	The space $C(X, Y)$ equipped with the compact-open topology
$C_d(X, Y)$	The space $C(X, Y)$ equipped with the uniform topology where d is a metric on Y
$C_f(X, Y)$	The space $C(X, Y)$ equipped with the fine topology
$C_g(X, Y)$	The space $C(X, Y)$ equipped with the graph topology
$C^*_\infty(X, Y)$	The set $C^*(X, Y)$ of all bounded functions in $C(X, Y)$ with the norm $\|\|\cdot\|\|_\infty$ where $\|\|f\|\|_\infty = sup\{\|\|f(x)\|\| : x \in X\}$
$H_p(X)$	The space $H(X)$ equipped with the point-open topology
$H_k(X)$	The space $H(X)$ equipped with the compact-open topology
$H_d(X)$	The space $H(X)$ equipped with the uniform topology where d is a metric on X
$H_f(X)$	The space $H(X)$ equipped with the fine topology
$B_d(f, \varepsilon)$	A basic open set in $C_d(X, Y)$ or $H_d(X)$
$B_f(h, \varepsilon)$	A basic open set in $C_f(X, Y)$ or $H_f(X)$
$B_g(f, \varepsilon)$	A basic open set in $C_g(X, Y)$

Introduction

The main objects of study in this monograph are three important topologies on the set $C(X, Y)$ of all continuous functions from a Tychonoff space X to a metric space Y, and the set $H(X)$ of all self-homeomorphisms on a metric space X. The set $C(X, Y)$ of all continuous functions from a Tychonoff space X to a metric space (Y, d) has a number of natural topologies like the point-open topology p, compact-open topology k, uniform topology d, fine topology f, and graph topology g. Although the point-open, compact-open, and graph topologies on $C(X, Y)$ can be defined for any topological space Y, in order to define the uniform topology on $C(X, Y)$, it is necessary for Y to have some uniform structure. On the other hand, the fine topology on $C(X, Y)$ can be defined only when Y is a metric space. The idea of topologizing $C(X, Y)$ arose from the notion of convergence of sequences of functions.

The spaces $C(X, Y)$ equipped with the point-open topology p, compact-open topology k, uniform topology d, fine topology f, and graph topology g are denoted, respectively, by $C_p(X, Y)$, $C_k(X, Y)$, $C_d(X, Y)$, $C_f(X, Y)$ and $C_g(X, Y)$.

In order to motivate the readers to have a closer look at this monograph with interest and attention, we give a short historic perspective for the function space topologies studied in this monograph.

The very concept of a function was not clear at the beginning of the nineteenth century. Nevertheless, the idea of pointwise convergence of a sequence of real-valued functions existed since the early days of calculus, particularly in the study of power and trigonometric series. But the uniform convergence of sequences of functions could not have been even imagined, until the concepts of convergent series and continuous function had been precisely described by Bolzano (1781–1848) and Cauchy (1789–1857). It suffices to mention that before them, the use of series without regard to convergence and divergence had led to a number of paradoxes and disagreements. In his 1817 publication, Bolzano had the correct notion of the conditions for the convergence of a sequence.

In 1821, while studying the limit of a convergent series of functions and the term-by-term integration of a series of continuous functions, Cauchy made some missteps and overlooked the need for uniform convergence. Fortunately soon after, Cauchy's errors came to the notice of Abel (1802–1829). In his 1826 paper, Abel gave a correct proof that sum of a uniformly convergent series of continuous functions is continuous in the interior of the interval of convergence. But he did not study the uniform convergence of a series of functions in its generality. The notion of uniform convergence and its subsequent importance were recognized in and for themselves by Stokes, a leading mathematical physicist of his time, and independently by Philipp L. Seidel in 1847–1848 and by Cauchy himself in 1853.

Actually Weierstrass (1815–1897) had the precise idea of uniform convergence with perfect clarity as early as 1842. But his work related to uniform convergence was first published much later in 1894. In fact, according to Stephen Willard ([107], p. 320), "In the last half of the 19th century, in the hands of Heine (1821–1881), Weierstrass, Riemann (1826–1866) and others, uniform convergence came into its own in applications to integration theory and Fourier series." Apparently, the work of Ascoli [13], Arzelà [12], and Hadamard [41] in the last two decades of the nineteenth century marked the beginning of what is known today as theory of function spaces. Loosely speaking, a topological space in which the points are functions is called a function space. The spaces of functions have been used since the late nineteenth century to form a framework in which convergence of sequences of functions could be studied.

The study of functions was continued in the twentieth century and resulted in the development of a new branch of mathematics known as the theory of functions of a real variable. Also a number of developments of nineteenth century, some of which have been mentioned earlier, crystallized in a new branch of geometry, now called topology. But for long, it was known as analysis situs. According to ([20], p. 162), "It is Riemann who should be considered as the creator of topology as of so many other branches of modern mathematics. He was the first person to attempt to formulate the notion of a topological space." But Weierstrass is considered to be the "father of modern analysis."

According to Morris Kline ([61], p. 1159), "Fréchet in 1906, stimulated by the desire to unify Cantor's theory of points of sets and treatment of functions as points of a space, launched the study of abstract spaces." Actually in 1913–1914, Hausdorff started developing the subject which is known today as general topology.

In view of the aforementioned facts, it will not be an exaggeration to say that the idea of topologizing the set of continuous functions from one topological space into another topological space arose from the notions of pointwise and uniform convergence of sequences of functions. While the topology of uniform convergence (called the uniform topology in this monograph) stems from the notion of a uniformly convergent sequence of functions, the topology of pointwise convergence (also called the point-open topology) stems from the older notion of a pointwise convergent sequence of functions. The topologies of pointwise convergence and uniform convergence are the first two function space topologies studied in the early years of general topology. The supremum metric topology, which is actually a topology of uniform convergence, was first studied by Fréchet [31] in 1906.

During his years as a high school teacher, Weierstrass discovered that any real-valued continuous function over a closed bounded interval in the real line can be expressed in that interval as the uniformly convergent limit of a sequence of polynomials. This result, better known as uniform approximation of real-valued continuous functions on a closed bounded interval, proved to be a strong and useful tool in classical analysis. In [95] and [96], M. H. Stone (1903–1989) enriched the theory of approximation of continuous functions by generalizing the aforesaid approximation theorem of Weierstrass substantially to real- or complex-valued functions having any compact Hausdorff space as domain.

To ensure more flexibility relating to uniform convergence in order to study the linear integral equations in 1911–12, E. H. Moore in [78] came up with the notion of relative uniformity of convergence or uniformity of convergence relative to a scale function. According to Moore, "for investigation in general analysis," "a more general notion" of uniform convergence was needed. In the definition of uniform convergence, he simply replaced the positive constant ε by a (strictly) positive function $\varepsilon(x)$ on the real line (that is, $\varepsilon(x) > 0$ for all $x \in \mathbb{R}$). Moore called these positive functions as the scale functions. This new type of uniform convergence relative to the scale functions coincides with the older, but more widely used uniform convergence, when one chooses simply positive constants in place of these scale functions. When one considers the sequences of continuous functions, these scale functions should be continuous as well.

We have already noted that in 1913–1914, Hausdorff started developing general topology. So it is quite expected that Moore, while talking about uniform convergence relative to the scale functions, did not talk about any compatible topology for this kind of uniform convergence on related function spaces. It took more than three decades for someone to put this special kind of uniform convergence in the right perspective of a function space topology. In 1948, Hewitt introduced in [44] the m-topology on the set $C(X)$ of all real-valued continuous functions on a topological space X. The convergence of a sequence of functions in $C(X)$ with respect to the m-topology is precisely the uniform convergence relative to the scale functions which are continuous on X and positive for all $x \in X$. In fact, in a footnote in [44], Hewitt noted that the m-topology for $C(\mathbb{R})$ was introduced by E. H. Moore in [78]. This m-topology has also been called in the literature the fine topology, the Whitney topology or Morse topology. But most of the general topologists usually prefer to call it the fine topology. According to Hewitt, "the topology of uniform convergence (u-topology) is most natural in considerations involving $C^*(X) = \{f \in C(X) : f \text{ is bounded}\}$, while the m-topology enjoys great advantages for the study of rings $C(X)$." In his 1991 paper [102], van Douwen, called the m-topology a natural generalization of the u-topology. In fact, the closure of an ideal in $C(X)$ with respect to the u-topology may not be an ideal, while the closure of an ideal in $C(X)$ with respect to the m-topology is again an ideal in $C(X)$, and consequently, every maximal ideal in $C(X)$ with respect to the m-topology is closed. In fact, it is one of the important reasons that many ring theorists have been interested in studying $C(X)$ equipped with the m-topology. See [3, 14, 15, 33, 34, 35, 68] and [90]. But in this monograph, we will not touch the algebraic study of the m-topology.

After [44], the m-topology was used by F. W. Anderson in approximating system of real-valued continuous functions in [3]. Also this fine topology has application in differential topology which can be found in [45] and [104]. Sometimes, more widely used topologies such as the compact-open topology and uniform topology (u-topology) are not strong enough to apply a function space to a given situation. A good example of this is the case of a “fine topology” on a function space in [101], in which the Baire space property of the function space is used to obtain certain kinds of embedding into infinite dimensional manifolds. Here, we should mention that usually the topologists and analysts consider the uniform topology to be quite nice as well as interesting for studying various topological and analytical properties of the spaces of continuous functions. In particular, this topology appears to be neither big nor small. So this topology on $C(X, Y)$ works as a sort of natural barrier between weaker and stronger topologies on $C(X, Y)$. While the point-open and compact-open topologies are weaker than the uniform topology, the m-topology, that is, the fine topology is stronger than the uniform topology. So it is quite natural to try to find out some other natural topologies on $C(X, Y)$ stronger than the uniform topology. The graph topology on $C(X, Y)$ is such a one.

In page 10 of [58], it has been noted, “Whatever a function is, its graph has an obvious definition as a set of ordered pairs. Moreover, there is no information about the function which cannot be derived from its graph. In brief, there is no reason why we should attempt to distinguish between a function and its graph.”

According to Naimpally, “beginning at least in 1936, (see [108]), there have been studies of graph topologies on (partial) functions.” But actually, it took another three decades before Naimpally himself put the graph topology firmly and visibly in the family of function space topologies in [82].

In his doctoral thesis [81], while dealing with the essential fixed points, Naimpally faced some difficulties in relation to some function spaces of non-continuous functions, more precisely in relation to almost continuous functions introduced by Stallings in [93]. A function f from a topological space X to another topological space Y is called almost continuous if for each open set U in $X \times Y$ containing $G(f)$, the graph of f, there exists a g in $C(X, Y)$ such that $G(g)$ is a subset of U. So an almost continuous function is one whose graph can be approximated by graphs of continuous functions. In order to tackle almost continuous functions efficiently, in 1964 Naimpally introduced a “new” function space topology, called the “graph topology,” in his doctoral thesis. Corresponding to each open set U in $X \times Y$, let $G_U = \{f \in C(X, Y) : G(f) \subseteq U\}$. Then, the collection $\{G_U : U \text{ is open in } X \times Y\}$ forms a base for the graph topology on $C(X, Y)$. Naimpally’s introductory work on the graph topology was published in [82].

In the introduction of the paper [18], the graph topology on $C(X, Y)$ has been described as follows.

If (Y,d) is a metric space, then a net (f_λ) in $C(X,Y)$ is uniformly convergent to f in $C(X,Y)$ meaning that given $\varepsilon > 0$, we have $d(f_\lambda(x), f(x)) < \varepsilon$ for all $x \in X$ and for all λ "sufficiently large" (the words "sufficiently large" have been used informally). Geometrically this means that $G(f_\lambda)$, the graph of f_λ, eventually lies in the open tube: $\{(x,y) : x \in X, d(y, f(x)) < \varepsilon\}$ surrounding the graph $G(f)$. But if we replace these tubular open sets in $X \times Y$ by the open neighborhoods V of $G(f)$, we get the graphical convergence, that is, the convergence of the net (f_λ) with respect to the graph topology. In particular, the graphical convergence is usually much stronger than the uniform convergence on $C(X,Y)$, when Y is a metric space.

There is an interesting fact that the graph topology on $C(X)$ can be linked with the famous Vietoris topology on the hyperspace $2^{X\times\mathbb{R}}$. As every multifunction from a topological space X to $\mathbb{R}$ can be viewed as a subset of $X \times \mathbb{R}$ by identifying it with its graph, the space of multifunctions with closed graphs can be considered as a subspace of the hyperspace $2^{X\times\mathbb{R}}$ of all closed subsets of $X \times \mathbb{R}$, and therefore, it can inherit hyperspace topologies as a subspace. Historically, there have been two well-known hyperspace topologies of particular importance: the Vietoris topology and the Hausdorff metric topology. The topology on $C(X)$ as a subspace of the space $CL(X,\mathbb{R})$ of all non-empty closed subsets of $X \times \mathbb{R}$ with the Vietoris topology is same as the graph topology on $C(X)$. Vietoris (1891–2002) introduced the famous topology named after him in 1922.

Since the sixties of the last century, many prominent mathematicians worked on both fine and graph topologies on spaces of continuous functions. The references to their works can be found in the list of references given at the end of this monograph. The first author of this monograph is one such mathematician who has made a contribution to the study of both the fine and graph topologies.

The goal of this monograph is to study topologically the uniform topology, fine topology, and graph topology on the space $C(X,Y)$, the set of all continuous functions from a Tychonoff space X to a metric space (Y,d); and on the space $H(X)$, the set of all self-homeomorphisms on a metric space (X,d). For a metric space (X,d), the fine and graph topologies on $H(X)$ coincide, (see [67, 70]). Also the space $H(X)$ is a group under the composition of mappings. But under the aforesaid topologies, it is a topological group only for the fine and graph topologies, (see [25, 70]), which happen to be same when X is a metric space. On the other hand, Arens has proved in [4] that whenever X is compact, or locally compact and locally connected, the space $H(X)$ with the compact-open topology is a topological group. Dijkstra in [26] has improved Arens's theorem to the situation that every point in X has a neighborhood that is a continuum (a compact connected set).

To be precise, the primary goal of this monograph is to study various topological properties of the spaces $C(X,Y)$ and $H(X)$ with the uniform, fine, and graph topologies, in terms of topological properties of X. We will mainly concentrate on metrizability, cardinal functions, countability properties, various kinds of completeness properties, connectedness, and compactness. More precisely, we would like to study

(A) The following properties of $C_d(X, Y)$, $C_f(X, Y)$, and $C_g(X, Y)$

   (i) Metrizability.
   (ii) Various cardinal functions such as character, density, weight, Lindelöf number, and cellularity and various kinds of countability properties such as first countability, separability, second countability, Lindelöf condition, and countable chain condition (CCC).
   (iii) Completeness properties such as complete metrizability, Čech-completeness, local Čech-completeness, sieve-completeness, partition-completeness, pseudo-completeness, and the property of being a Baire space.
   (iv) Connectedness and path connectedness.
   (v) Compactness.

(B) The following properties of $H(X)$

   (i) Separability, connectedness, and path connectedness of the space $H(X)$ under the uniform and fine topologies.
   (ii) First countability and metrizability of the space $H(X)$ under the fine topology.

In addition to the properties listed above, in case of $H(\mathbb{R}^n)$, that is, the homeomorphism spaces on Euclidean spaces, we use three different compatible metrics on the space $\mathbb{R}^n$ to define three different uniform topologies on $H(\mathbb{R}^n)$. We study the relation of these topologies among themselves and with various other topologies on $H(\mathbb{R}^n)$. We also discuss the relation of various topologies on the space $H(\mathbb{R})$ with various product topologies such as the Tychonoff product topology, box product topology, and semi-box product topology on the space $\mathbb{R}^\omega$, where ω is the first infinite ordinal.

The entire work of this monograph has been presented in six chapters.

In Chap. 1, we give the definitions of various function space topologies on the space $C(X, Y)$ such as the point-open topology p, compact-open topology k, uniform topology d, fine topology f, and graph topology g and study the relations between these function space topologies. A number of examples are given to illustrate these relations. Finally, we discuss the dependence of the uniform and fine topologies upon the choice of a compatible metric on the range space Y.

In Chap. 2, we study the metrizability and various kinds of completeness properties of the fine and graph topologies on the space $C(X, Y)$, the set of all continuous functions from a Tychonoff space X to a metric space Y.

Chapter 3 is devoted to the study of cardinal functions on the spaces $C(X)$ equipped with the uniform, fine, and graph topologies. We also give characterization of various countability properties of the uniform, fine, and graph topologies on the space $C(X, Y)$.

In Chap. 4, we study the connectedness and some related algebraic properties of the uniform, fine, and graph topologies on the space $C(X, Y)$, the set of all continuous functions from a Tychonoff space X to a normed linear space $(Y, ||\cdot||)$, where we consider Y as a metric space with the metric induced by the norm $||\cdot||$. We show that these spaces are in general not connected, and in that case, we determine the components and path components of these spaces. We also study the necessary and sufficient conditions for these spaces to be connected.

In Chap. 5, we do a brief study of the compact subsets of $C(X, Y)$ under the uniform, fine, and graph topologies and prove the Stone–Weierstrass approximation theorem in detail.

In Chap. 6 of this monograph, we study extensively the space $H(X)$, the set of all homeomorphisms from a metric space X onto itself, where $H(X)$ has either the uniform topology or fine topology. In particular, we study the separability and connectedness of the space $H(X)$ with the uniform and fine topologies. Here, we recall that for a metric space X, the fine and graph topologies on $H(X)$ coincide, (see [67, 70]). Also for $X = \mathbb{R}^n$, three different natural compatible metrics are used to generate three different uniform topologies on $H(\mathbb{R}^n)$. These three homeomorphism spaces are shown to be not homeomorphic to each other for $n > 1$ and are also compared to $H(\mathbb{R}^n)$ with the fine, point-open, and compact-open topologies. Then, we investigate the relation between the space $H(\mathbb{R})$ with various function space topologies and the various product topologies on the set $\mathbb{R}^\omega$. At the end of this chapter, we give the components and path components of the uniform and fine topologies on the spaces of homeomorphisms on the Euclidean spaces $\mathbb{R}^n$.

In this monograph, we use the following conventions. Unless otherwise mentioned, every topological space X is always assumed to be a Tychonoff space and every normed linear space Y is assumed to be a non-trivial normed linear space over the field of real numbers. Also we define the function $0_X : X \to Y$ by $0_X(x) = \mathbf{0}$ for all $x \in X$, where $\mathbf{0}$ denotes the zero element of the space Y. The symbols $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{N}$, respectively, denote the sets of all real numbers, rational numbers, and natural numbers, and ω denotes the first infinite ordinal. We shall always assume that $\mathbb{R}$ and its subsets have the usual topology, unless otherwise mentioned. For a subset A of X, the closure of A is denoted $\overline{A}$. Moreover, in any metric space, $B(x, \varepsilon)$ denotes the open ball centered at x with radius $\varepsilon > 0$ and $\overline{B}(x, \varepsilon)$ denotes the closed ball centered at x with radius $\varepsilon > 0$. Also a basic open set in $C_d(X, Y)$ or $H_d(X)$ is denoted by $B_d(h, \varepsilon)$; in $C_f(X, Y)$ or $H_f(X)$, it is denoted by $B_f(h, \varepsilon)$, and in $C_g(X, Y)$, it is denoted by $B_g(h, \varepsilon)$. If A and B are two subsets of X, then $A \backslash B$ denotes the set $\{x \in A : x \notin B\}$. In particular, for any subset A of X, $X \backslash A$ denotes the complement of A. For topological spaces S, and T that have the same underlying set, we use the notations $S = T$, $S \leq T$, and $S < T$ to mean that, respectively, the topology on S is equal to the topology on T, the topology on S is coarser than or equal to the topology on T, and the topology on S is strictly coarser than the topology on T.

Finally, in the monograph, we take one numbering for the Definitions, one for the Examples, one for the Remarks, and another one for the Propositions, Lemmas, Theorems, and Corollaries, each numbering being restricted to its own chapter.

Note: For the historical perspective given in this chapter, the authors have taken substantial help from [21] and [61].