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Path Coupling and Aggregate Path Coupling

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To Evgenia and Susie

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Preface

The coupling method is known as one of the few purely probabilistic techniques in mathematics. In combination with other methods, the coupling has been an effective tool in solving a variety of mathematical problems. For example, coupling can be used for proving limit theorems, or uniqueness of limit measures. In the theory of interacting particle systems, coupling is used for proving main invariance results (see [38] and [39]). The history of the coupling method dates back to the work of Doeblin [18] as documented by Lindvall [40], where a facsimile of parts of [18] is included in the Epilogue. The renewed interest in the coupling method was sparked by the reemergence of the mixing times [36]. Besides being used for proving the convergence results for Markov processes, the coupling method is also used for estimating the speed of convergence, characterized by the mixing times.

The theory of mixing times addresses a fundamental question that lies at the heart of statistical mechanics. How quickly does a physical system relax to equilibrium? A related problem arises in computational statistical physics concerning the accuracy of computer simulations of equilibrium data. One typically carries out such simulations by running Glauber dynamics or the closely related Metropolis algorithm, in which case the theory of mixing times allows one to quantify the running time required by the simulation.

An important question driving the work in the field is the relationship between the mixing times of the dynamics and the equilibrium phase transition structure of the corresponding statistical mechanical models. The *path coupling* method introduced by Bubley and Dyer [7] is a powerful tool in the theory of mixing times of Markov chains in which rapid mixing can be proved by showing that the mean coupling distance contracts between all neighboring configurations of a minimal path connecting two arbitrary configurations. Many results for statistical mechanical models that exhibit a continuous phase transition were obtained by a direct application of the standard path coupling method.

For models that exhibit a first-order/discontinuous phase transition, the standard path coupling method did not work. Thus, the path coupling method needed to be extended for the cases when the mean coupling distance did not contract for some of the neighboring configurations. This extension, developed in [35, 34, 31], is referred to as *aggregate path coupling*. The aggregate path coupling method extends the use of the path coupling technique in the absence of contraction of the mean coupling distance between all neighboring configurations of a statistical mechanical model. In this monograph, we show how to combine aggregate path coupling and large deviation theory [22] to determine the mixing times of a large class of statistical mechanical models, including those that exhibit a first-order phase transition. Our primary objective is to characterize the assumptions required to apply the method of aggregate path coupling.

In this monograph, the complete theory of aggregate path coupling is presented. While many of the results were first introduced in original research papers, here they are presented in a unifying theory, in greater generality, and with complete and precise background, so that the book can serve as a stand-alone reference for the theory of path coupling and aggregate path coupling. The monograph is organized as follows. Chapter 1 begins with an overview on mixing times, coupling, and maximal coupling. There, we introduce synchronized maximal coupling of three random variables and examine its applicability in Lemma 1.11 and Corollary 1.13, which we then use to rigorously justify the path coupling method. Chapter 1 ends with Section 1.8, where Theorem 1.19 that encompasses the main steps in the aggregate path coupling method is proven.

A class of statistical mechanical models considered in this monograph is defined in Chapter 2. There, Glauber dynamics is introduced, and distinct types of phase transition are discussed in Section 2.3. The two types of phase transition, continuous and first-order, are rigorously defined in Chapter 3, which covers large deviation theory and equilibrium macrostates for the statistical mechanical models in Chapter 2.

Chapter 4 provides an example of successful use of path coupling (i.e., identifying the parameter region of fast mixing) in the Curie-Weiss model, which exhibits continuous phase transition. In the chapters following Chapter 4, we define and characterize the aggregate path coupling method for three classes of models. First, in Chapter 5, we cover the simpler setting, where the macroscopic quantity for the model is one dimensional. Then in Chapter 6, we generalize the ideas of Chapter 5 to a large class of statistical mechanical models with macroscopic quantities that are higher dimensional, including the mixing time results in Section 6.4 of Chapter 6 for a Glauber dynamics that converges to the so-called generalized Potts model on the complete graph. Finally, in Chapter 7, we develop the aggregate path coupling theory for the case where the underlying graph of the model is bipartite graph $K_{n,n}$.

Parts of this monograph are based on the collaboration [31] with José Cerda Hernández. We would like to acknowledge the continued support and encouragements for this work we received from Richard S. Ellis and Ed Waymire. We would like to thank Jon Machta for providing us with valuable advice on how to introduce phase transition in this monograph. Many people attended our presentations and provided their feedback and perspectives. Among them were Thomas M. Liggett, Amir Dembo, Robert M. Burton, Enrique Thomann, Zhen-Qing Chen, Anatoly Yambartsev, Sourav Chatterjee, and many others. We thank Bruno Barbosa, a doctoral student at Oregon State University, for pointing out a number of inaccuracies in an earlier draft. Finally, we would like to thank the anonymous referees for providing us with important comments and motivating remarks. This work was supported in part by the NSF award DMS-1412557.

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