

# Lecture Notes in Mathematics

2210

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Mathematics in Vietnam goes back to ancient times. Over five hundred years ago in Hanoi the name of Luong Thê Vinh, an expert in geometry, was inscribed on a stele of honor in Văn Miếu.

Over sixty years ago, the Việt Minh published a geometry textbook written by Hoàng Tuy for schools in the liberated zones, a rare case of a guerrilla press publishing a mathematics book!

Founded in 2010 after the award of the Fields Medal to Ngô Bảo Châu, the Vietnam Institute for Advanced Study in Mathematics VIASM officially opened in Hanoi in 2011, aiming to become a leading research center where Vietnamese mathematicians can develop projects and nurture young talent. Ngô Bảo Châu, one of the initiators, became the scientific director in 2011.

VIASM engages in traditional research areas of pure and applied mathematics, as well as applying mathematics in other fields such as physics, computer science, biology and economics. The main activity of the Institute is the organization of research groups to conduct high quality research programs and projects. International and Vietnamese scientists in the same field gather and work together at the Institute. VIASM organizes conferences, workshops, seminars on topics associated with research groups working at the Institute, special schools for mathematics students, short-term training courses for mathematics teachers and common activities to disseminate scientific knowledge to the public and support the application of mathematics in socio-economic development.

The VIASM subseries of the Lecture Notes in Mathematics publishes high quality original articles or survey papers on topics of current interest. They are based on lectures delivered in special periods organized at the Vietnam Institute for Advanced Study in Mathematics (VIASM). With the agreement of the Editors of the LNM Series, and as a temporary arrangement, the first volumes are not subjected to the strict LNM rules of coherency for multi-author volumes.

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# Commutative Algebra and its Interactions to Algebraic Geometry

VIASM 2013–2014

 Springer

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# Preface

This collection of notes is based on four lectures given during the programme *Commutative Algebra* at the Vietnam Institute of Advanced Study in Mathematics in the winter semester 2013–2014. The lectures provide introductions to recent research topics in Commutative Algebra, which are related to Algebraic Geometry and other fields. The topics were chosen to represent different aspects of the use of the basic tools of Commutative Algebra. The notes are mainly self-contained, with the hope that students with advanced backgrounds in algebra can get through and absorb different techniques and ideas in Commutative Algebra before settling on concrete research problems. They can also be used separately as courses for graduate students, depending on the level and interest of the students.

The first lecture, by M. Brodmann, offers an introduction to the theory of rings of differential operators and their modules, also known as Weyl algebras and  $D$ -modules. These concepts relate Non-commutative Algebra and Commutative Algebra with Algebraic Geometry and Analysis in a very appealing way. The lecture presents this theory from the viewpoint of Commutative Algebra and is aimed at an audience having only a basic background in Commutative Algebra. The main feature is therefore not to explain everything about Weyl algebras and  $D$ -modules, but only the relevant aspects which are directly related to Commutative Algebra, such as the characteristic variety via the theory of filtered algebras and modules. The last part also contains some recent results on the stability, deformation and defining equations of the characteristic variety. The material is developed systematically and is accompanied by examples and exercises. These notes are well suited for an undergraduate course.

The second lecture, by J. Elias, is a short introduction to the theory of inverse systems and its application in the classification of Artinian Gorenstein rings. The classification of Artinian rings (rings of finite length) up to analytic isomorphism is a basic problem in Commutative Algebra and Algebraic Geometry. This problem is even open for Artinian Gorenstein rings, when the ring is an injective module over itself. Inverse systems provide an important tool in Commutative Algebra, establishing a beautiful correspondence between Artinian Gorenstein quotient rings and certain polynomials via derivations. The notes give a thorough introduction to

the theory of injective modules and inverse systems and show how to use these tools to classify Artinian Gorenstein rings and to compute their Betti numbers. The presented material combines several basic techniques of Commutative Algebra and could be used for a graduate course.

The third lecture, by R.M. Miró-Roig, is on the complexity of the structure of projective varieties. This complexity can be measured by the representation type, which is the dimension and the number of families of indecomposable arithmetically Cohen–Macaulay sheaves (i.e. sheaves without intermediate cohomology) on the underlying variety. This is a fascinating topic of Algebraic Geometry, which requires an advanced background in Commutative Algebra. The notes cover the basic facts on this and related subjects such as moduli spaces of sheaves, liaison theory, minimal resolutions and Hilbert schemes of points. Many interesting results are presented on arithmetically Cohen–Macaulay sheaves and bundles having natural extremal algebraic properties, and several examples of varieties of wild representation type are given. The exposition is self-contained and features numerous open problems and promising ideas for further investigation. It may serve as a graduate course in Algebraic Geometry.

The last lecture, by M. Morales, addresses a classical problem of both Commutative Algebra and Algebraic Geometry, namely, how many equations are needed to define an algebraic variety set-theoretically. This seemingly simple problem is wide open even for toric varieties, which are given parametrically by monomials. The notes provide an extensive survey on this problem in the case of simplicial toric varieties, which are defined by the property that the exponents of the parametrizing monomials span a simplicial complex. One can use arithmetical and combinatorial tools (semigroups, lattices) to obtain satisfactory results for large classes of simplicial toric varieties. The material is presented in a systematic way and can easily be followed by any reader with some basic background in Commutative Algebra. These notes are recommended as a first course for anyone who wants to see the interaction between algebra, combinatorics and geometry. They can be used as a starting point for graduate studies in Commutative Algebra.

Hanoi, Vietnam  
14 October 2017

Nguyen Tu CUONG  
Le Tuan HOA  
Ngo Viet TRUNG

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