

# Logic, Epistemology, and the Unity of Science

Volume 42

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Enrico Martino

# Intuitionistic Proof Versus Classical Truth

The Role of Brouwer's Creative Subject  
in Intuitionistic Mathematics

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*Così è (se vi pare)*  
*So it is (if you think so)*

Luigi Pirandello

*To Livia*

# Preface

I was stimulated by Göran Sundholm to collect some of my old papers on Intuitionism, which turn back to the eightieths and the ninetieths of the past century. In those years, there was a lively philosophical debate between classical and intuitionistic logicians and mathematicians. I was especially interested in the works of Dummett, Troelstra, van Dalen, Sundholm, Veldman and others.

A main peculiarity of my research is a deep analysis of the main tenets of the pioneers of Intuitionism: Brouwer and Heyting. In particular, it is analysed Heyting's explanation of the intuitionistic meaning of logical constants and Brouwer's idealisation of the creative subject as grounds of intuitionistic truth.

Besides, it is upheld the importance of the role of certain imaginary acts of choice performed by an ideal agent for explaining the notion of reference not only in intuitionistic but also in classical logic.

A crucial question, discussed in the work, is the following: to what extent succeeds the intuitionistic perspective to avoid the classical realistic notion of truth? My answer is that a form of realism is hidden in the idealisation of Brouwer's creative subject. In fact, in order to exploit the role of the creative subject, we need to think him as if he were a real being: the mere imagine of him in our mind would not be able to perform the actions required by his role.

Some papers of the present collection are written together with Daniele Giaretta and Gabriele Usberti.

I'm grateful to Göran Sundholm for his interest in my work on Intuitionism. I thank the Springer editor Shahid Rahman. Besides of thank Ali Mohammed, Stephen O'Reilly and Nisha Keeran. A particular thank to my colleague Vittorio Morato for preparing the manuscript in LATEX.

Padua, Italy  
June 2017

Enrico Martino

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