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Applied and Numerical Harmonic Analysis

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Compressed Sensing and Its Applications

Third International MATHEON Conference 2017

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ANHA Series Preface

The *Applied and Numerical Harmonic Analysis (ANHA)* book series aims to provide the engineering, mathematical, and scientific communities with significant developments in harmonic analysis, ranging from abstract harmonic analysis to basic applications. The title of the series reflects the importance of applications and numerical implementation, but richness and relevance of applications and implementation depend fundamentally on the structure and depth of theoretical underpinnings. Thus, from our point of view, the interleaving of theory and applications and their creative symbiotic evolution is axiomatic.

Harmonic analysis is a wellspring of ideas and applicability that has flourished, developed, and deepened over time within many disciplines and by means of creative cross-fertilization with diverse areas. The intricate and fundamental relationship between harmonic analysis and fields such as signal processing, partial differential equations (PDEs), and image processing is reflected in our state-of-the-art *ANHA* series.

Our vision of modern harmonic analysis includes mathematical areas such as wavelet theory, Banach algebras, classical Fourier analysis, time-frequency analysis, and fractal geometry, as well as the diverse topics that impinge on them.

For example, wavelet theory can be considered an appropriate tool to deal with some basic problems in digital signal processing, speech and image processing, geophysics, pattern recognition, biomedical engineering, and turbulence. These areas implement the latest technology from sampling methods on surfaces to fast algorithms and computer vision methods. The underlying mathematics of wavelet theory depends not only on classical Fourier analysis, but also on ideas from abstract harmonic analysis, including von Neumann algebras and the affine group. This leads to a study of the Heisenberg group and its relationship to Gabor systems, and of the metaplectic group for a meaningful interaction of signal decomposition methods. The unifying influence of wavelet theory in the aforementioned topics illustrates the justification for providing a means for centralizing and disseminating information from the broader, but still focused, area of harmonic analysis. This will be a key role of *ANHA*. We intend to publish with the scope and interaction that such a host of issues demands.

Along with our commitment to publish mathematically significant works at the frontiers of harmonic analysis, we have a comparably strong commitment to publish major advances in the following applicable topics in which harmonic analysis plays a substantial role:

Antenna theory	*Prediction theory*
Biomedical signal processing	*Radar applications*
Digital signal processing	*Sampling theory*
Fast algorithms	*Spectral estimation*
Gabor theory and applications	*Speech processing*
Image processing	*Time-frequency and time-scale analysis*
Numerical partial differential equations	*Wavelet theory*

The above point of view for the *ANHA* book series is inspired by the history of Fourier analysis itself, whose tentacles reach into so many fields.

In the last two centuries, Fourier analysis has had a major impact on the development of mathematics, on the understanding of many engineering and scientific phenomena, and on the solution of some of the most important problems in mathematics and the sciences. Historically, Fourier series was developed in the analysis of some of the classical PDEs of mathematical physics; these series were used to solve such equations. In order to understand Fourier series and the kinds of solutions they could represent, some of the most basic notions of analysis were defined, e.g., the concept of "function." Since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantor's set theory was also developed because of such uniqueness questions.

A basic problem in Fourier analysis is to show how complicated phenomena, such as sound waves, can be described in terms of elementary harmonics. There are two aspects of this problem: first, to find, or even define properly, the harmonics or spectrum of a given phenomenon, e.g., the spectroscopy problem in optics; second, to determine which phenomena can be constructed from given classes of harmonics, as done, for example, by the mechanical synthesizers in tidal analysis.

Fourier analysis is also the natural setting for many other problems in engineering, mathematics, and the sciences. For example, Wiener's Tauberian theorem in Fourier analysis not only characterizes the behavior of the prime numbers, but also provides the proper notion of spectrum for phenomena such as white light; this latter process leads to the Fourier analysis associated with correlation functions in filtering and prediction problems, and these problems, in turn, deal naturally with Hardy spaces in the theory of complex variables.

Nowadays, some of the theory of PDEs has given way to the study of Fourier integral operators. Problems in antenna theory are studied in terms of unimodular trigonometric polynomials. Applications of Fourier analysis abound in signal processing, whether with the fast Fourier transform (FFT), or filter design, or the adaptive modeling inherent in time-frequency-scale methods such as wavelet theory. The coherent states of mathematical physics are translated and modulated Fourier transforms, and these are used, in conjunction with the uncertainty principle, for dealing with signal reconstruction in communications theory. We are back to the raison d'être of the *ANHA* series!

College Park, MD, USA

John J. Benedetto
Series Editor
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Preface

Compressed sensing is an efficient technique to measure and reconstruct high-dimensional signals. The key idea of this method is that high-dimensional signals usually admit a lower dimensional structure in the sense that they have a sparse representation in a basis or a frame.

Since the publication of the first papers in 2006, compressed sensing has established itself as an independent field of research, and its mathematical foundations are nowadays well understood. Along the way, the area has benefitted and was driven by its interdisciplinarity. Indeed, compressed sensing is located at the interface of applied mathematics and engineering with applications in communication theory, imaging sciences, optics, radar technology, sensor networks, and tomography.

In this spirit, two MATHEON conferences entitled "Compressed Sensing and its Applications" were held in December 2013 and December 2015 at the Technische Universität Berlin. These brought together experts from a variety of research areas including electrical engineering, mathematics, biology, chemistry, computer science, or material scientists. Both workshops were supported by the Matheon, which is a research center in Berlin for "Mathematics for Key Technologies", as well as by the German Research Foundation (DFG).

Due to the overwhelming success of the previous workshops, the editors of this volume organized a third edition of the conference series in 2017. In addition to the established field of compressed sensing, we decided to open the conference up to applications of deep learning in data science as we expected substantial overlap of these methods and ideas and those prevalent in compressed sensing. Overall, we welcomed 140 participants from 12 countries with an immense variety of different backgrounds leading to fruitful and inspiring discussions.

This volume contains a selection of contributions from speakers of this conference. It is aimed at a broad readership including graduate students and researchers in the areas of mathematics, computer science, and engineering. We believe it is also accessible to researchers working in any other field requiring methodologies for data science. Hence, this volume can be used both as a state-of-the-art monograph on applications of compressed sensing and as a textbook for graduate students. Here is a brief outline of the contents of each chapter.

Chapter "An Introduction to Compressed Sensing" provides an introduction as well as a self-contained overview of the main results on the theory and applications of compressed sensing. Chapters "Quantized Compressed Sensing: A Survey", "On Reconstructing Functions from Binary Measurements", and "Classification Scheme for Binary Data with Extensions" explore the role of quantization in data science applications. More specifically, Chapter "Quantized Compressed Sensing: A Survey" gives a survey on quantized compressed sensing, Chapter "On Reconstructing Functions from Binary Measurements" analyses reconstruction of functions from binary measurements and Chapter "Classification Scheme for Binary Data with Extensions" introduces a classification algorithm from binary measurements. Chapters "Generalization Error in Deep Learning", "Deep Learning for Trivial Inverse Problems", and "Oracle Inequalities for Local and Global Empirical Risk Minimizers" discuss aspects of the area of machine learning. To be precise, Chapter "Generalization Error in Deep Learning" presents a survey on theoretical results on the generalization error in machine learning techniques. Chapter "Deep Learning for Trivial Inverse Problems" studies the feasibility of deep learning techniques to solve inverse problems. Chapter "Oracle Inequalities for Local and Global Empirical Risk Minimizers" establishes oracle inequalities for empirical risk minimization. Chapter "Median-Truncated Gradient Descent: A Robust and Scalable Nonconvex Approach for Signal Estimation" presents a variation of gradient descent with applications in traditional compressed sensing as well as machine learning. In the final chapter of this book a practical example of compressed sensing in single pixel imaging is presented.

This conference certainly would not have been possible without the support of dedicated volunteers, and we gratefully acknowledge the help of all members of the Applied Functional Analysis Group at the Technische Universität Berlin Tiep Dovan, Katharina Eller, Axel Flinth, Ansgar Freyer, Ingo Gühring, Martin Genzel, Ali Hashemi, Anja Hedrich, Sandra Keiper, Héctor Andrade Loarca, Jan Macdonald, and Stephan Wäldchen.

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Acronyms

ADC	Analog-to-digital converter
ADMM	Alternating direction method of multipliers
AMP	Approximate message passing
BOS	Bounded orthonormal system
BP	Basis pursuit
BPDN	Basis pursuit denoising
CoSaMP	Compressive sampling matching pursuit
CS	Compressed sensing
DCT	Discrete cosine transform
DFT	Discrete Fourier transform
EM	Expectation maximization
FFT	Fast Fourier transform
FISTA	Fast iterative soft thresholding algorithm
HTP	Hard thresholding pursuit
IHT	Iterative hard thresholding
IRLS	Iteratively reweighted least squares
ISTA	Iterative soft thresholding algorithm
LASSO	Least-absolute shrinkage selection operator
LP	Linear program
MGF	Moment generating function
MSE	Mean squared error
NSP	Null space property
OMP	Orthogonal matching pursuit
QCBP	Quadratically constrained basis pursuit
RIC	Restricted isometry constant
RIP	Restricted isometry property
SOCP	Second-order cone program
SOS	Sum of squares