Lecture Notes in Physics

Volume 946

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Magnetohydrodynamics and Fluid Dynamics: Action Principles and Conservation Laws

Springer

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ISSN 0075-8450 ISSN 1616-6361 (electronic)
Lecture Notes in Physics
ISBN 978-3-319-72510-9 ISBN 978-3-319-72511-6 (eBook)
https://doi.org/10.1007/978-3-319-72511-6

Library of Congress Control Number: 2017963310

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

The motivation for the present book originated in the quest to understand wave-wave interactions in magnetohydrodynamics (MHD) in a non-uniform background flow (this process is sometimes referred to as wave mixing in the solar wind and in cosmic ray modified shocks). The variational approach to WKB wave propagation in a non-uniform background plasma flow was developed by Dewar (1970). My initial aim was to understand linear, non-WKB wave propagation in the solar wind. The problem of wave mixing has also been identified as an important process in the evolution of turbulence and Alfvénic fluctuations in the solar wind (e.g. Zhou and Matthaeus 1990a,b; Zank et al. 2012). Waves in non-uniform flows also play an important role in Lagrangian averaged Euler-Poincaré equations (LAEP equations) of wave-mean flow interactions and the so-called alpha model of turbulence (e.g. Holm 2002).

Another motivation for the book was to understand the elegant non-canonical Hamiltonian formalism for MHD and fluids developed by Morrison and Greene (1980, 1982), Holm and Kupershmidt (1983a,b) and Marsden et al. (1984). The connection between a Clebsch variable action principle for MHD and the non-canonical Poisson bracket of Morrison and Greene (1980, 1982) and the Clebsch variational approach is developed by Zakharov and Kuznetsov (1997) (see also Zakharov and Kuznetsov (1971) for the canonical form of Hamilton's equations for MHD using Clebsch variables). In particular the work of Padhye and Morrison (1996a,b) shows the connection between Noether's second theorem and the conservation of potential vorticity in ideal fluid mechanics and MHD, due to the fluid relabelling symmetry of the equations (see also Salmon (1982, 1988) for an account of the fluid relabelling symmetry in ideal fluids). The fluid relabelling symmetries are due to the invariance of the action, in which the Lagrangian fluid labels can change (i.e. there are transformations or maps of the fluid labels onto new fluid labels that are diffeomorphisms) but the usual physical variables remain invariant. There are relationships between the fluid relabelling symmetries and the Casimirs of the non-canonical MHD Poisson bracket, which are explored in the present lecture notes.

Yet another motivation for the book is applications of topological methods in fluid dynamics and MHD. In the book we give examples of magnetic helicity conservation (e.g. Woltjer 1958; Kruskal and Kulsrud 1958; Berger and Field 1984; Finn and Antonsen 1985; Moffatt 1969, 1978; Moffatt and Ricca 1992) in solar physics and in solar wind physics. In Chap. 2, Sect. 2.5, we describe the investigation of Torok et al. (2010, 2014) on the evolution of the twist and writhe components of magnetic helicity in the evolution of the kink instability for solar magnetic flux ropes, and its role in coronal mass ejections (CMEs). Other applications to the magnetic helicity of the Parker interplanetary, Archimedean spiral magnetic field, to nonlinear Alfvén waves in the solar wind, and the MHD topological soliton solutions are described in Chap. 6.

Conservation laws obtained by Lie dragging advected invariants in magnetohydrodynamics (MHD) and gas dynamics or hydrodynamics (HD) were investigated by Moiseev et al. (1982), Sagdeev et al. (1990), Tur and Yanovsky (1993), Volkov et al. (1995), Kats (2001, 2003, 2004) and Webb et al. (2014a). The ten Galilean, Lie point symmetries of the action give rise to the energy conservation, momentum conservation, angular momentum and centre of mass conservation laws, via Noether's first theorem. The advected invariants are due to fluid relabelling symmetries, or diffeomorphisms associated with the Lagrangian map. There are different classes of geometrical quantities that are advected or Lie dragged with the flow. Examples are the entropy S (a 0-form) and the conservation of the magnetic flux ($\mathbf{B} \cdot d\mathbf{S}$ which is an invariant advected two-form), moving with the flow (i.e. Faraday's equation). Advected invariants are obtained by using the Euler-Poincaré approach to Noether's second theorem. Some of the invariants are important in topological fluid dynamics and MHD. We discuss different variants of helicity including kinetic helicity, cross helicity, magnetic helicity, Ertel's theorem and potential vorticity, the Hollman invariant and the Godbillon Vey invariant. Lie dragged invariants or Cauchy invariants play an important role in describing the dynamics of vortex and magnetic field lines in ideal hydrodynamics and MHD (e.g. Kuznetsov and Ruban 1998, 2000; Kuznetsov 2006; Besse and Frisch 2017).

The multi-symplectic and multi-momentum approach to Hamiltonian systems was originally developed by de Donder (1930) and Weyl (1935). They studied generalized Hamiltonian mechanics in which the Lagrangian $L = L(\mathbf{x}, \varphi^i, \partial\varphi^i/\partial x^\mu)$ where $x^\mu (1 \leq \mu \leq n)$ are the independent variables and φ^k $(1 \leq k \leq m)$ are the dependent variables. For the case where $n \geq 2$ one can define multi-momenta $\pi_i^\mu = \partial\varphi^i/\partial x^\mu$ corresponding to each x^μ (in the usual Hamiltonian formulation $x^0 = t$ is the evolution variable). The multi-symplectic approach has been developed in field theory in the search for a more covariant form of Hamiltonian mechanics (in the usual Hamiltonian formulation, there is only one evolution variable). Bridges et al. (2005, 2010), Marsden and Shkoller (1999), Hydon (2005) and Cotter et al. (2007) describe multi-symplectic systems. Our aim is to present both Eulerian and Lagrangian variational principles for ideal fluids and MHD obtained by, e.g. Newcomb (1962), Holm and Kupershmidt (1983a,b), Dewar (1970) and Webb et al. (2005a,b, 2014a,b). Both Eulerian and Lagrangian multi-symplectic forms of the equations can be obtained. In this book we concentrate on the Eulerian multi-symplectic form of the equations (the Lagrangian, multi-symplectic ideal fluid equations are described by Webb (2015) and Webb and Anco (2016)). The multi-symplectic Noether's theorem and symplecticity and pullback conservation laws are obtained. Nonlocal conservation laws, for a non-barotropic equation of state for the fluid, in which the time integral of the temperature back along the fluid path plays an important memory role, are obtained (see also Mobbs (1981) for similar conservation laws for helicity in non-barotropic fluids). Yahalom (2016a, 2017a,b) explores the physical and topological meaning of the non-barotropic cross helicity and cross helicity per unit magnetic field flux, using a Clebsch potential formulation (see also Webb and Anco 2017). The connection of the multi-symplectic approach with Cartan's theory of differential equations using differential forms is developed. A potential vorticity type conservation law is derived for MHD using Noether's second theorem.

The motivation is to provide both local and nonlocal conservation laws of the fluid and MHD equations that give insight into the physics. Conservation laws are useful for the testing numerical codes and reveal new aspects of the physics (e.g. nonlocal conservation laws associated with potential symmetries and fluid relabelling symmetries, reveal the time history of the fluid elements can play an important role in understanding fluid vorticity). For example, the baroclinic effect leads to the creation of vorticity in fluids (e.g. in tornadoes), but the corresponding nonlocal conservation law for fluid helicity is not usually discussed. Casimirs (i.e. quantities with zero Poisson bracket with other functionals of the physical variables) are important in describing the stability of steady flows and equilibria. The knowledge of new conservation laws is important in fusion plasmas, space plasmas, fluid dynamics and atmospheric physics. New conservation laws are also important in mathematics in elucidating the symmetries responsible for the conservation laws (e.g. Lie pseudo groups are most likely related to fluid relabelling symmetries).

What Is Not Included in the Book

The abstract geometrical mechanics aspects of fluid mechanics and MHD are not developed in the present approach. Detailed descriptions of the geometrical mechanics approach to the theory are described in Marsden et al. (1984), Marsden and Ratiu (1994), Holm et al. (1998) and Holm (2008a,b). Holm and Kupershmidt (1983a,b), Marsden et al. (1984) and Holm et al. (1998) describe the role of semi-direct product Lie algebras and Lie groups inherent in the non-canonical Poisson bracket of Morrison and Greene (1980, 1982). Morrison (1982) gives a direct algebraic method to derive the Jacobi identity. Olver (1993) uses the variational complex to develop methods to check if a given co-symplectic differential operator used to define the Poisson bracket is a Hamiltonian operator (i.e. the bracket is skew symmetric and satisfies the Jacobi identity). Chandre et al. (2012, 2013) and Chandre (2013) derived Dirac brackets for MHD to obtain well-behaved brackets that satisfy the Jacobi identity. Bridges et al. (2010) use the variational bi-complex to describe multi-symplectic systems. The analysis of Lie symmetries of differential equation systems using Lie's algorithm (e.g. Bluman and Kumei 1989; Olver 1993; Ovsjannikov 1962, 1982; Ibragimov 1985; Bluman et al. 2010) can be used to derive analytical solutions of the equations. We do not study conservation laws and symmetries for special and general relativistic MHD (see, e.g. Lichnerowicz 1967; Beckenstein and Oron 1978; Bekenstein 1987; Anile 1989; Achterberg 1983; D'Avignon et al. 2015). Pshenitsin (2016) has derived infinite classes of conservation laws for incompressible viscous MHD by using the so-called direct method developed by Anco and Bluman (see, e.g. Bluman et al. 2010). This method of determining conservation laws is illustrated for the case of the KdV equation in Chap. 4. However, we have not used this method to derive MHD conservation laws in the present book.

We discuss topological invariants in fluids and plasmas, using Lie dragged invariants in ideal fluids and MHD (see, e.g. Arnold and Khesin 1998; Berger and Field 1984; Berger 1999a,b; Moffatt and Ricca 1992; Besse and Frisch 2017 for detailed analysis). The papers by Kuznetsov and Ruban (1998, 2000) and Kuznetsov et al. (2004) give an account of vortex lines and magnetic field lines, using a mixed Eulerian and Lagrangian approach, which shows how one may resolve the degeneracy of the non-canonical Poisson brackets, by using Weber transformations and Lagrangian representations of the equations. They also show how the Hasimoto transformation arises from their analysis. Euler potential representations of the magnetic field and its use in fusion and space plasmas are another large area of research not covered in our treatment (see, e.g. Stern (1966) for applications in space plasmas, and Boozer (2004) in fusion plasmas).

Recent work by Webb (2015) and Webb and Anco (2016) on Lagrangian, multi-symplectic fluid mechanics and work on MHD gauge field theory by Webb and Anco (2017) are omitted from the present exposition. It is worth noting that Calkin (1963) developed a version of gauge field theory for a polarized version of MHD. Both Calkin (1963) and Webb and Anco (2017) identified the gauge symmetry responsible for the magnetic helicity conservation law in MHD. These developments lie beyond the scope of the present book.

Acknowledgements

The material in this book originated in a series of papers on MHD with my colleagues at the IGPP, University of California Riverside (2002–2008), and at the CSPAR at the University of Alabama in Huntsville (G.P. Zank, Q. Hu, B. Dasgupta, N.P. Pogorelov and J.F. McKenzie). I am indebted to Professor Darryl Holm (Mathematics Dept., Imperial College London) and Professor Phil. Morrison (Department of Physics and Institute of Fusion Studies of the University of Texas at Austin) for discussions on the mathematics and physics of conservation laws and symmetries, Poisson brackets, Casimirs, Euler-Poincaré equations, multi-symplectic systems,

Lie symmetries and Noether's theorems in MHD and fluid dynamics. I am indebted to G.P. Zank for suggesting to write a book on MHD and fluids and for discussions of the compressible, incompressible and reduced MHD equations and symmetries of the equations. I am indebted to Stephen Anco for discussions on some of the more obscure conservation laws of the MHD and fluid equations and the use of the direct method to obtain conservation laws for equation systems, which are not necessarily associated with variational principles (e.g. Cheviakov 2014; Cheviakov and Oberlack 2014; Pshenitsin 2016). I am grateful to Dr. Q. Hu (who drew many of the figures) for discussions on magnetic helicity and magnetic clouds, and to Dr. B. Dasgupta for his detailed knowledge of magnetic fields in plasma physics (e.g. the Kamchatnov MHD topological soliton and chaotic versus integrable magnetic field lines). I am indebted to E.D. Fackerell and C.B.G. McIntosh for their lectures on Lie symmetries and differential equations, whilst at Monash University in the period 1974–1977. I also acknowledge discussions on fluid relabelling symmetries with Nikhil Padhye and discussions with R.B. Sheldon on the importance of nonlocal conservation laws. The errors are mine.

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January 16, 2018

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