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A Second Course in Topos Quantum Theory

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Vorrei dedicare questo libro ai mie Zii, Gina Ricciardi e Celestino Cruciani per la stima e l'affetto che hanno sempre mostrato verso di me ed il mio lavoro.

Contents

Notation and Terminology

For us, 'C*-algebra' always means 'unital C*-algebra'. Likewise, our $*$-homomorphisms are always assumed to be unital, unless noted otherwise (as in the proof of Theorem 13.4.1). This already applies to the following index of our notation, which lists the conventions for our most commonly used mathematical symbols:

W, X, Y, Z	Compact Hausdorff spaces
$\mathbf{1}, \ldots, \mathbf{4}$	A compact Hausdorff on the corresponding number of points, where we write e.g. $\mathbf{4} = \{0, 1, 2, 3\}$
w, x, y, z	Points in a compact Hausdorff space
f, g, h, k	Continuous functions between compact Hausdorff spaces
$\square, \bigcirc, \mathbb{T}$	Unit square, unit disk and unit circle, considered as compact subsets of $\mathbb{C}$
A, B	C*-algebras or piecewise C*-algebras (Definition 3.1.5)
M_n	the C*-algebra of $n \times n$ matrices with entries in $\mathbb{C}$
$\alpha, \beta, \gamma, \nu, \tau$	Normal elements in a C*-algebra, or (more generally) $*$-homomorphisms of the type $C(X) \to A$
ζ	A $*$-homomorphism or piecewise $*$-homomorphism of the type $A \to B$
$\mathfrak{a}, \mathfrak{b}$	Self-action of a piecewise C*-algebra (Definition 13.4.1) or a piecewise group (Definition 13.5.3)

The normal part of a C*-algebra A is

$$\mathbb{C}(A) := \{ \alpha \in A \mid \alpha\alpha^* = \alpha^*\alpha \}.$$

We also think of it as the set of 'A-points' of $\mathbb{C}$. More generally, for $A \in \mathsf{C^*alg}_1$ and a closed subset $S \subset \mathbb{C}$, we also write

$$S(A) := \{ \alpha \in \mathbb{C}(A) \mid \mathrm{sp}(\alpha) \subseteq S \}$$

for the set of normal elements with spectrum in S, and similarly $S(\zeta) : S(A) \to S(B)$ for the resulting action of a $*$-homomorphism $\zeta : A \to B$ on these elements. For example, $\mathbb{R}(A)$ denotes the self-adjoint part of a C*-algebra, and similarly $\mathbb{T}(A)$ is the unitary group. This sort of notation may be familiar from algebraic geometry, where the set of A-points of a scheme S (over a ring A) is denoted $S(A)$. We also use the standard notation $C(X)$ for the $\mathbb{C}$-valued continuous functions on a space X. Unfortunately, this is very similar notation despite being different in nature.

We work with the following categories:

CHaus	Compact Hausdorff spaces with continuous maps
CGHaus	Compactly generated Hausdorff spaces with continuous maps
$\mathsf{C^*alg_1}$	C*-algebras with $*$-homomorphisms
$\mathsf{cC^*alg_1}$	Commutative C*-algebras with $*$-homomorphisms
$\mathcal{V}(\mathcal{H})$	Context category (Definition 1.1.1)
$\underline{\Omega}$	Sub-object classifier (Definition 1.3.1)
S	This generally indicates a sieve (Definition 1.3.2)
$\delta^o(\hat{P})$	Outer daseinisation of projector $\hat{P}$ (Definition 2.2.2)
cHa	Complete Heyting algebra (Definition 2.1.5)
$Sub_{cl}(\underline{\Sigma})$	Set of all clopen sub-objects of $\underline{\Sigma}$ (Definition 2.2.1)
$\mathfrak{S}_V$	Isomorphism of complete Boolean algebras (Definition 2.2.1 and Eq. (3.4.3))
$\underline{\Sigma}$	Spectral presheaf over $\mathcal{V}(\mathcal{H})$ (Definition 3.1.1)
$\mathsf{pC^*alg_1}$	Piecewise C*-algebras (Definition 3.1.5) with piecewise $*$-homomorphisms (Definition 3.1.6)
Sets	Category of sets
$\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\mathrm{op}}}$	Topos of presheaves over $\mathcal{V}(\mathcal{H})$
$\underline{\Sigma}^{\mathcal{A}}$	Spectral presheaf over a C*-algebra $\mathcal{A}$ (Definition 3.1.1)
ucC*	Category of unital abelian C^*-algebras and unital $*$-homomorphisms
KHaus	Category of compact Hausdorff spaces and continuous maps
$\langle \Phi, \mathcal{G}_\phi \rangle$	Automorphism of spectral presheaf (Definition 3.2.1)
$F_{\hat{A}}$	Flow on the spectral presheaf (Definition 3.4.1)
$\overline{F}^{-1}(t)$	General flow (Definition 3.4.2)
$\overline{F}_{\hat{A}}^{-1}(t)$	Flows induced by unitaries (Definition 3.4.3)
μ	Measure on the state-space $\underline{\Sigma}$ (Definition 3.4.4)
$\underline{CP}$	Presheaf of classical probability measures on $\underline{\Sigma}^{\mathcal{N}}$ (Definition 3.4.5)
$\overline{F}_{\hat{A}}$	Flow on $\Gamma\underline{CP}$ induced by one-parameter group of unitaries (Definition 3.4.7)
$C_{\mathcal{N}}(F)$	Filter in $P(\mathcal{N})$ (Eq. (4.1.4))
$g_{\hat{A}}$	Antonymous function of $\hat{A}$ (Definition 4.1.2)
$f_{\hat{A}}$	Observable function of $\hat{A}$ (Definition 4.1.2)
$\delta^i(\hat{P})$	Inner daseinisation (Definition 4.1.4)
$\underline{\breve{\delta}(\hat{A})}$	Physical quantity associated with $\hat{A}$ (Definition 4.1.16)
$\overline{\delta^i(\hat{A})_V}$	Gelfand transform associated with $\delta^i(\hat{A})_V$ (Corollary 4.1.1)
$\overline{\delta^o(\hat{A})_V}$	Gelfand transform associated with $\delta^o(\hat{A})_V$ (Corollary 4.1.2)

$\underline{\mathfrak{w}}^{\lvert\psi\rangle}$	Pseudo-state (Definition 4.2.1)
$\overline{\mathbb{R}}$	Extended reals (Definition 5.1.1)
E	Spectral family (Definition 5.1.1)
$o^{\hat{A}}$	q-Observable function associated with $\hat{A}$ (Definition 5.1.4)
$SA(\mathcal{N})$	Set of self-adjoint operators affiliated with a von Neumann algebra $\mathcal{N}$
$SF(\overline{\mathbb{R}}, P(\mathcal{N}))$	Set of extended, right-continuous spectral families of $P(\mathcal{N})$
$QO(P(\mathcal{N}), \overline{\mathbb{R}})$	Set of all abstract q-observable functions
$a^{\hat{A}}$	q-Antonymous function associated with $\hat{A}$ (Definition 5.3.1)
$\tilde{C}^A$	A cumulative distribution function (CDF) of a random variable A (Definition 5.5.1)
C^A	An extended cumulative distribution function (ECDF) of a random variable A (Definition 5.5.1)
q^A	Quantile function of A (Eq. (5.5.1))
$\overline{C}^A$	Lattice valued CDF (Definition 5.5.3)
$Sub(\mathcal{N})$	Set of all von Neumann subalgebras of $\mathcal{N}$ (Definition 6.1.1)
$AbSub(\mathcal{N})$	Set of all abelian subalgebras of $\mathcal{N}$ (Definition 6.1.1)
$FAbSub(\mathcal{N})$	Set of all abelian subalgebras of $\mathcal{N}$ containing only finitely many projections (Definition 6.1.1)
$Sub(Proj(\mathcal{N}))$	Poset of subalgebras of $Proj(\mathcal{N})$ ordered by subset inclusion (Definition 6.1.2)
$BSub(Proj(\mathcal{N}))$	Poset of Boolean subalgebras of $Proj(\mathcal{N})$ ordered by subset inclusion (Definition 6.1.2)
$FBSub(Proj(\mathcal{N}))$	Poset of finite Boolean subalgebras of $Proj(\mathcal{N})$ ordered by subset inclusion (Definition 6.1.2)
PG	Set of commutative Lie subalgebras of $L(G)$ (Section 14.1)
$\underline{R}$	Presheaf of quantizations over PG (Definition 14.1.1)
$\underline{I}$	Pre-quantization presheaf (Definition 14.1.3)
$\mathcal{O}(X)$	Category of open subsets of the topological space X
J	A Grothendieck topology seen as a function on a category (Definition 7.1.5)
$(\mathcal{C}, J)$	Site, consisting of a category $\mathcal{C}$ and a Grothendieck topology J
K	Basis for a Grothendieck topology (Definition 7.1.11)
Loc	Category of locales with continuous maps
Spaces	Category of topological space with continuous maps
pt(X)	Points of a local X
$\underline{A}$	Internal C^*-Algebra in a topos (Definition 9.2.1)
CStar	Category of internal unital C^*-algebras, together with internal unital *-homomorphism
KRegLoc	Category of compact regular locales
Σ	A first order signature (Definition 10.2.1)
M	A Σ-structure (Definition 10.3.1)
Σ-**Str**(τ)	Category of Σ-structures and Σ-structure homomorphisms
$\vec{x}.t$	Term in context (Definition 10.3.2)
$\vec{x}.\phi$	Formula in context (Definition 10.3.3)

$\mathbb{T}$	Theory over a sequent Σ (Definition 10.2.5)
$\mathbb{T}$-**Mod**(τ)	Category of models for a given theory $\mathbb{T}$
Poset	Category of partially ordered sets and monotone functions
Topos	Category whose objects are topos and whose morphisms are geometric morphisms
$\overline{\mathcal{A}}$	Presheaf representing the internal C^*-algebra (Definition 11.1.1)
$\overline{\Sigma}_{\overline{\mathcal{A}}}$	Internal spectrum of $\overline{\mathcal{A}}$
$\Sigma_{\downarrow}$	Topological space associated with $\overline{\Sigma}_{\overline{\mathcal{A}}}$ (Definition 11.2.1)
$I : \mathcal{A}_{sa} \to \mathbb{R}$	Probability integral (Definition 11.3.3)
$\Sigma_{\uparrow}$	Alternative definition of state space (Definition 11.4.1)
$\overline{\delta}(\hat{A})^{-1}$	Covariant daseinisation map (Definition 11.4.2)
$[\hat{A} \in (p, q)]_1$	Covariant proposition (Definition 11.4.3)
$\overline{\mathbb{R}}_l$	Internal lower reals (Definition 11.5.1)
$\overline{\mathbb{R}}_u$	Internal upper reals (Definition 11.5.2)
$\underline{\mathbb{R}^{\leftrightarrow}}$	Quantity valued object (Definition 12.3.1)
$\underline{\breve{\mathbb{R}}}^{\leftrightarrow}$	Alternative quantity value object (Definition 12.4.1)
$\mathsf{aC^*alg}_1$	Almost C*-algebras (Definition 13.4.1) with almost ∗-homomorphisms (Definition 13.4.2)
Grp	Groups with group homomorphisms
pGrp	Piecewise groups (Definition 13.5.1) with piecewise group homomorphisms (Definition 13.5.2)
aGrp	Almost groups (Definition 13.5.3) with almost group homomorphisms (Definition 13.5.4)