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Boundary Value Problems with Global Projection Conditions

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Preface

Boundary value problems (BVPs) for elliptic and other types of partial differential equations belong to the classical areas of mathematical analysis. Prototypes are the Dirichlet or the Neumann problem for the Laplace operator, and practically every textbook on PDEs treats problems of that type. However, it is by no means obvious whether (or that) the operators which are involved in the solvability process (such as Green's function, or potential operators) have a pseudo-differential structure, or how many other elliptic boundary value problems can be posed for the Laplacian. Another question is whether an elliptic differential operator admits (Shapiro–Lopatinskii)-elliptic boundary conditions at all, for instance, the Cauchy–Riemann operator in a smooth domain in the complex plane, or Dirac operators on a manifold with smooth boundary. The more we look in this direction, the more obscure the notion of ellipticity of a BVP becomes. And if there are also singularities on the boundary (even of a moderate complexity, such as conical points or edges), substantial difficulties arise in the search for a natural approach to reflect basic solvability properties.

The present text is devoted to developing general concepts of ellipticity of BVPs. First we introduce necessary tools on pseudo-differential operators. We define ellipticity on an open C^∞ manifold and construct parametrices within the algebra of standard pseudo-differential operators. Then we pass to Toeplitz operators on a closed compact C^∞ manifold based on pseudo-differential operators and pseudo-differential projections. Other essential topics are operators with operator-valued symbols with twisted symbol estimates, where we establish basic results. We also present some material on pseudo-differential operators on manifolds with conical exit to infinity, especially manifolds modelled on infinite cylinders (or half-cylinders). Then we study elliptic BVPs of Shapiro–Lopatinskii type in the framework of Boutet de Monvel's calculus [10]. In particular, we give a K-theoretic explanation of the existence of such conditions, cf. Atiyah and Bott [3], Harutyunyan and Schulze [23, Subsection 3.3.4], and elucidate the pseudo-differential structure of parametrices. After that we formulate a Toeplitz analogue of the algebras of pseudo-differential operators with the transmission property at the boundary. Such an operator algebra, first introduced in [47] (see also a more detailed version in [48]), is built for similar reasons as other pseudo-differential algebras, namely, so as to contain all "standard" elliptic boundary value problems for differential operators and to be closed under the construction of parametrices of elliptic elements. While the space of BVPs for differential operators with Shapiro–Lopatinskii (SL)-elliptic conditions just generates the above-mentioned algebra of BVPs with the transmission property at the boundary, cf. [10] or [34], [19], the Toeplitz analogue is designed to include the parametrices of Dirac operators with global projection conditions (especially APS-conditions in the sense of Atiyah, Patodi, and Singer [4, 5, 6]) as well as elliptic BVPs for geometric and other elliptic differential operators, with conditions of SL-elliptic or global projection (GP)-elliptic type. More precisely, every elliptic differential (and then also pseudo-differential operator with the transmission property at the boundary) on a smooth compact manifold with boundary belongs to the algebra, and, as we shall see, every such operator admits elliptic boundary conditions of that kind.

In that sense the Toeplitz algebra of BVPs unifies the concept of elliptic conditions of SL- and GP-elliptic type. Ellipticity in this context is equivalent with the Fredholm properties in the respective scales of spaces (standard Sobolev spaces in the SL case, spaces of Hardy type in the GP case). The 2×2 block matrices contain the above-mentioned algebra of Toeplitz operators on the boundary as a subalgebra, and hence also the Fredholm property of elliptic operators between Hardy-type Sobolev spaces is equivalent with the respective GP-ellipticity. In order to make the machinery transparent, we provide a concise introduction to the Boutet de Monvel algebra of BVPs with the transmission property at the boundary. In this framework, we consider cutting and pasting of elliptic BVPs and analogues of index formulas of Agranovich and Dynin, and we analyse the pseudo-differential nature of projections of Calderón–Seeley type.

Ellipticity on manifolds with boundary and approaches to treating solvability near the boundary constitute a prominent field of PDEs, geometric analysis, and index theory. Numerous papers and monographs are devoted to special cases and explicit computations, see, for instance, Booss-Bavnbek and Wojciechowski [8], Grubb and Seeley [20], Savin and Sternin [37], and the work of many other authors, in particular, joint work [53] with Seiler on elliptic complexes of BVPs in GP-framework, and references therein. Additional references and results can be found in the monograph [29] of Nazaikinskij et al.

Another part of this text studies a Toeplitz analogue of the edge algebra, see, in particular, the articles of Schulze and Seiler [52, 54]. The original edge pseudo-differential algebra was introduced in [43] as a calculus that contains all edge-degenerate differential operators on a manifold with edge, together with the parametrices of elliptic elements. The ellipticity first refers to an analogue of Shapiro–Lopatinskii conditions, i.e., a bijectivity condition for an operator-valued principal symbol structure which contains also trace and potential operators with respect to the edge, a substitute of the former boundary. In order to keep the material self-contained, we briefly outline the basic parts of the edge calculus. Again there is a topological obstruction to the existence of such edge conditions, and we complete the algebra by edge conditions of global projection type to an algebra referred to as the Toeplitz analogue of the edge algebra. It contains Shapiro–Lopatinskii elliptic edge conditions as a special case. Similarly as in the calculus of BVPs, we obtain the Fredholm property in spaces on the boundary that are derived from standard Sobolev spaces and a subsequent pseudo-differential projection. We show that elliptic conditions of global projection type exist for arbitrary edge-degenerate elliptic operators in the top left corner.

We then pass to a special case of the edge calculus, namely, BVPs on a manifold with smooth boundary, cf. the article of Schulze and Seiler [51]. Clearly, all results of the general edge calculus developed before remain true. However, regarding a manifold with boundary as a special manifold with edge allows us to single out a specific subclass of pseudo-differential operators which are more in the focus of boundary value problems, namely, operators with standard symbols rather than edge-degenerate ones obtained by restriction of pseudo-differential operators of an ambient open manifold containing the considered embedded manifold with smooth boundary. It is by no means evident that these operators generate a subcalculus of the general edge algebra. However, we show that this is indeed the case, and we obtain an approach of BVPs without (or with) the transmission property at the boundary which is much more general than the calculus of [10]. Note in this connection that the theory of Vishik and Eskin also treats BVPs without the transmission property, cf. [62], [63] and Eskin's book [14], but the edge calculus approach is rather different, and it produces an algebra. Nevertheless, [14] contains an algebra of pseudo-differential operators on the half-line which became later on an important ingredient of the edge symbolic calulus. This part of the development is also commented in detail in the monograph [45], see also Rempel and Schulze [35]. BVPs without the transmission property at the boundary are of interest also in connection with mixed and transmission problems, cf. Harutyunyan and Schulze [23], Wong [56], Chang et al. [11]. Here we mainly focus on the feature that elliptic boundary problems with global projection conditions can be studied for similar reasons as in the corresponding Toeplitz variant of the general edge calculus. But it remains a remarkable effect that there are relevant subalgebras which are closed under the construction of parametrices.

Contents

Introduction

Boundary value problems are important topics in the analysis of partial differential equations. The motivation comes from physics and wide areas of the applied sciences, but also from index theory, geometry, and other fields of pure mathematics. Similarly to the case of open manifolds, where ellipticity plays a crucial role in understanding solvability properties for many types of equations, e.g., also parabolic ones, ellipticity on a manifold with boundary is not only interesting on its own right, but also for large classes of more general problems, where the control of phenomena up to the boundary is a specific aspect. Moreover, a (say, smooth) boundary can be interpreted as a special geometric singularity, namely, as an edge with the inner normal as the model cone of a corresponding wedge, here a collar neighbourhood of the boundary. It turns out that the analysis of elliptic BVPs yields methods and insight on ellipticity on manifolds with singularities, and many ideas in this field can be read off from the case of boundary value problems. Conversely, it turns out that the ideas for analysing equations on manifolds with singularities, especially for cones and wedges, shed a new light on the structure of solvability of BVPs, e.g., mixed problems, for instance, the Zaremba problem for the Laplacian, with mixed conditions of Dirichlet and Neumann type which have a jump on an interface of the boundary, cf. [23], or [11].

A smooth manifold X with boundary ∂X can be interpreted as a stratified space, i.e., a disjoint union of smooth open manifolds

$$X = s_0(X) \cup s_1(X),$$

with $s_0(X) := \operatorname{int} X = X \setminus \partial X$ and $s_1(X) := \partial X$. Parallel to the stratification of X, symbolised by the sequence of strata

$$s(X) = \big(s_0(X), s_1(X)\big),$$

the operators A under consideration have a principal symbol hierarchy

$$\sigma(A) = \big(\sigma_0(A), \sigma_1(A)\big) \tag{0.1}$$

with $\sigma_i(A)$ being associated with $s_i(X)$, $i = 0, 1$, where $\sigma_0(A)$ is also called the principal interior symbol and $\sigma_1(A)$ the principal boundary symbol of A. Later on, in Section 2.3, we prefer to write $(\sigma_\psi, \sigma_\partial)$ rather than (σ_0, σ_1).

If A is a differential operator of order $\mu \in \mathbb{N}$ $(= \{0, 1, 2, \dots\})$ written locally in coordinates $x \in \mathbb{R}^n$ in the form

$$A = \sum_{|\alpha| \le \mu} a_\alpha(x) D_x^\alpha$$

with smooth coefficients, where X close to the boundary is identified with the half-space $\overline{\mathbb{R}}^n_+ = \{x \in \mathbb{R}^n : x_n \ge 0\}$, we have

$$\sigma_0(A)(x, \xi) = \sum_{|\alpha| = \mu} a_\alpha(x) \xi^\alpha \tag{0.2}$$

and

$$\sigma_1(A)(x', \xi') = \sum_{|\alpha| = \mu} a_\alpha(x', 0) \big(\xi', D_{x_n}\big)^\alpha, \tag{0.3}$$

where $x = (x', x_n)$, $\xi = (\xi', \xi_n)$. While (0.2) is interpreted as a smooth scalar function on $T^*X \setminus 0$, the cotangent bundle of X with the zero section removed, (0.3) is a smooth operator-valued function on $T^*(\partial X) \setminus 0$. The action of the symbol (0.3) can be regarded in standard Sobolev spaces on $\mathbb{R}_+$, namely,

$$\sigma_1(A)(x', \xi') : H^s(\mathbb{R}_+) \to H^{s-\mu}(\mathbb{R}_+), \tag{0.4}$$

for any $s \in \mathbb{R}$. The meaning of (0.1) in the pseudo-differential set-up will be explained below. If the operator A is elliptic with respect to σ_0, i.e., the symbol (0.2) does not vanish for $\xi \ne 0$, the operators (0.4) form a family of Fredholm operators for $s - \mu > -1/2$. Ellipticity of A with respect to (0.1) should include an invertibility condition on (0.4) for $\xi' \ne 0$. However, the operator family (0.4) is bijective only in exceptional cases (in the pseudo-differential set-up). For a differential operator A the boundary symbol $\sigma_1(A)$ is surjective but not injective. In order to have invertibility we should complete the Fredholm family (0.4) to a family of isomorphisms

$$\sigma_1(\mathcal{A})(x', \xi') := \begin{pmatrix} \sigma_1(A) \\ \sigma_1(T) \end{pmatrix} (x', \xi') : H^s(\mathbb{R}_+) \to \begin{matrix} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N_2} \end{matrix} \tag{0.5}$$

by an extra operator family $\sigma_1(T)(x', \xi') : H^s(\mathbb{R}_+) \to \mathbb{C}^{N_2}$ that maps the kernel of $\sigma_1(A)(x', \xi')$ isomorphically to $\mathbb{C}^{N_2}$. Globally $\mathbb{C}^{N_2}$ is interpreted as the fibre of a vector bundle G over $T^*(\partial X) \setminus 0$; in general, this bundle is not trivial. In addition, to interpret $\sigma_1(T)$ as the boundary symbol of an operator

$$T : H^s(X) \to H^{s-\mu}(\partial X, J_2) \tag{0.6}$$

for some smooth complex vector bundle J_2 over ∂X, we have to require that

$$G = \pi^*_{\partial X} J_2, \tag{0.7}$$

i.e., the above-mentioned G has to be the pull-back of such a J_2 under the canonical projection $\pi_{\partial X} : T^*(\partial X) \setminus 0 \to \partial X$. As for the shift μ of smoothness in (0.6), we are free to impose any real number; for convenience we took $\mu = \operatorname{ord} A$, the order of A. The property (0.7) is a topological condition on the behavior of $\sigma_0(A)$ close to the boundary, see [3] and Section 3.2 below, necessary and sufficient for the existence of a Fredholm operator of the form

$$\mathcal{A} = \begin{pmatrix} A & K \\ T & Q \end{pmatrix} : \begin{matrix} H^s(X) \\ \oplus \\ H^s(\partial X, J_1) \end{matrix} \to \begin{matrix} H^{s-\mu}(X) \\ \oplus \\ H^{s-\mu}(\partial X, J_2) \end{matrix} \tag{0.8}$$

belonging to the Boutet de Monvel's calculus, in the general pseudo-differential case also containing non-trivial entries K, Q and another smooth complex vector bundle J_1. The Fredholm property of (0.8) is equivalent to the σ_0-ellipticity of A together with the bijectivity of the boundary symbol

$$\sigma_1(\mathcal{A})(x', \xi') = \begin{pmatrix} \sigma_1(A) & \sigma_1(K) \\ \sigma_1(T) & \sigma_1(Q) \end{pmatrix} (x', \xi') : \begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ \left(\pi^*_{\partial X} J_1\right)_{x', \xi'} \end{matrix} \to \begin{matrix} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \left(\pi^*_{\partial X} J_2\right)_{x', \xi'} \end{matrix} \tag{0.9}$$

for every $(x', \xi') \in T^*(\partial X) \setminus 0$ and sufficiently large s. The relation (0.9) is a block matrix generalisation of (0.5), meaningful in the pseudo-differential case, and formulated in global form. The bundles J_1, J_2 over ∂X appear at the same time as soon as both $\ker \sigma_1(A)(x', \xi')$ and $\operatorname{coker} \sigma_1(A)(x', \xi')$ are non-trivial.

The operators K, T, Q are said to satisfy the (pseudo-differential analogue of the) Shapiro–Lopatinskii condition with respect to A if (0.9) is a family of bijections. In Chapter 3 we will provide more details, in particular, on boundary conditions when the above-mentioned topological obstruction does not vanish. A general answer was first given in [47]. The corresponding extension of Boutet de Monvel's calculus to a Toeplitz calculus is presented here in Chapter 4. Special geometric differential operators have been studied before, including extra global boundary conditions that guarantee a finite Fredholm index, see the work of Atiyah, Patodi, and Singer [4, 5, 6] and of many other authors.

In Chapter 5 we study in detail the case of BVPs for differential operators. First, in Section 5.1, we establish general cutting and pasting constructions and compare the indices of elliptic operators on a closed manifold M with the indices of elliptic BVPs on submanifolds M_+, M_- with common smooth boundary Y that subdivide M as $M_+ \cup M_-$ with $M_+ \cap M_- = Y$. In Section 5.2 we study an extension of the concept of spectral BVPs to arbitrary elliptic differential operators of any order, following joint work [30], [28] with Nazaikinskij, Savin, Sternin and Shatalov. Section 5.3 is devoted to further observations on Calderón–Seeley projections.

In Part II we consider elliptic operators on a manifold with edge, here with edge conditions rather than boundary conditions, again under the unifying goal of understanding Shapiro–Lopatinskii and global projection edge conditions within a Toeplitz analogue of the edge calculus. This material refers to a paper of Schulze and Seiler [52], and again we focus on the operator algebra aspect and the construction of parametrices of elliptic elements within this calculus.

Part III is devoted to the case of Toeplitz calculus where the underlying space is a smooth manifold with boundary. This corresponds to the case where the model cone transverse to the edge, the boundary, is equal to $\overline{\mathbb{R}}_+$, the inner normal. We refer to the fact that the calculus contains all interior classical symbols, not necessarily with the transmission property, which are smooth up to the boundary. Such a subcalculus is much more general than that of Part I with the transmission property, though here we realize operators in weighted edge spaces. In Section 3.2 we briefly outline other approaches to symbols without the transmission property and make interesting observations, suggested by several applications. Moreover, we give a number of modifications and extensions of the calculus, such as to the truncation quantization, already occurring both in Vishik and Eskin's work as well in the Boutet de Monvel calculus, see also the joint paper [50, 51] with Seiler.

It is interesting to study operators on more general singular spaces with boundary, e.g., when the boundary itself has conical singularities, edges or higher corners. However, this problem seems to be open, at least as far as the operator algebra approach with global projection conditions is concerned. A "final" calculus answer should be fitted in the general strategy of establishing operator algebras with symbol hierarchies on stratified spaces with boundary, analogously to the program outlined in [49].

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