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Quantum Triangulations

Moduli Space, Quantum Computing, Non-Linear Sigma Models and Ricci Flow

Second Edition

Springer

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Preface to the First Edition

The above illustration shows a variant woodcut printer's device on the verso last leaf of a rare sixteenth-century edition of Plato's *Timaeus* (*Divini Platonis Operum a Marsilio Ficino tralatorum, Tomus Quartus. Lugduni, apud Joan Tornaesium M.D.XXXXX*). The printer's device to the colophon shows a medallion with a tetrahedron in the center and the motto round the border: *Nescit Labi Virtus* (Virtue Cannot Fail). This woodcut beautifully illustrates the role of the perfect shape of the tetrahedron in classical culture. The tetrahedron conveys such a profound impression of stability as to be considered an epitome of virtue, unfailingly inspiring us with the depth and elegance of its shape. In the course of history, the geometry of the tetrahedron, of the Platonic solids, and more generally of the highly symmetrical discrete patterns one encounters in nature and art has always been connected with some of the more sophisticated aspects of mathematics and physics of the day. From Plato's *Timaeus* to Piero della Francesca's *Libellus de Quinque Corporibus Regularibus* to Pacioli's *De Divina Proportione* up to Kepler's *Harmonices Mundi*, there have always been attempts to involve the Platonic solids and their many variants to provide mathematical models for the physical universe. What makes these shapes perfectly irresistible to many mathematicians and physicists, both amateur and professional, is culturally related to their long-standing role in natural philosophy but also to the recondite fact that the geometry of these discrete structures often points to unexpected connections between very distinct aspects of mathematics and physics. And modern theoretical physics has drawn even further attention to the latter property.

Indeed, polyhedral manifolds, the natural generalization of Platonic solids, play quite a distinguished role in such settings as Riemann moduli space theory, strings and quantum gravity, topological quantum field theory, condensed matter physics, and critical phenomena. The motivation for such a wide spectrum of applications goes beyond the observation that polyhedral manifolds provide a natural discrete analogue of the smooth manifolds in which a physical theory is framed. Rather, it is often a consequence of an underlying structure, only apparently combinatorial, which naturally calls into play nontrivial aspects of representation theory, complex analysis, and topology in a way that makes manifest the basic geometric structures of the physical interactions involved. In spite of these remarks, one has to admit that, in almost all existing literature, the role of triangulated manifolds remains merely a convenient discretization of the physical theory, a grab bag of techniques that are computationally rather than conceptually apt to disclose the underlying physics and geometry. The restriction to such a computational role may indeed be justified by the physical nature of the problem, as is often the case in critical statistical field theory, but sometimes it is not. This is the discriminating criterion motivating these lecture notes, since, in the broad panorama the theory offers, the relation between polyhedral surfaces, Riemann moduli spaces, noncritical string theory, and quantum computing emerges as a clear path probing the connection between triangulated manifolds and quantum physics to the deepest levels.

Chapter 1 is devoted to a detailed study of the geometry of polyhedral manifolds and in particular triangulated surfaces. This subject, which may be considered a classic, has recently seen a flourishing of many new results of great potential impact in the physical applications of the theory.

Here the focus is on results which are either new or not readily accessible in the standard repertoire. In particular, we discuss from an original perspective the structure of the space of all polyhedral surfaces of a given genus and their stable degenerations. In such a framework, and within the whole landscape of the space of polyhedral surfaces, an important role is played by the conical singularities associated with the Euclidean triangulation of a surface. We provide a detailed analysis of the geometry of these singularities and introduce the associated notion of cotangent cones, circle bundles, and the attendant Euler class on the space of polyhedral surfaces. This is a rather delicate point which appears in many guises in quantum gravity and string theory and which is related to the role that Riemann moduli spaces play in these theories. Not surprisingly, the Witten–Kontsevich model lurks in the background of our analysis, and some of the notions we introduce may well serve to illustrate, from a more elementary point of view, the often very technical definitions that characterize this subject.

We turn in Chap. 2 to the formulation of a powerful dictionary between polyhedral surfaces and complex geometry. It must be noted that, in both the mathematical and the physical applications of the theory, the connection between Riemann surfaces and triangulations typically emphasizes the role of ribbon graphs and the associated metric. The conical geometry of the polyhedral surface is left aside and seems to play no significant role. This attitude can be motivated by Troyanov's basic observation that the conformal structure does not see the conical singularities of a polyhedral surface. However, this gives a rather narrow perspective on the much wider role that the theory has to offer. Thus, we relate a polyhedral surface to a corresponding Riemann surface by fully taking into account its conical geometry. This connection is many-faceted and exploits a vast repertoire of notions, ranging from complex function theory to algebraic geometry. We start by defining the barycentrically dual polytope associated with a polyhedral surface and discuss the geometry of the corresponding ribbon graph.

By adapting to our case an elegant version of Strebel's theorem provided by Mulase, we explicitly construct the Riemann surface associated with the dual polytope. This brings us directly to the analysis of Troyanov's singular Euclidean structures and to the construction of the bijective map between the moduli space $\mathfrak{M}_{g,N_0}$ of Riemann surfaces (M, N_0) with N_0 marked points, decorated with conical angles, and the space of polyhedral structures. In particular, the first Chern class of the line bundles naturally defined over $\mathfrak{M}_{g,N_0}$ by the cotangent space at the ith marked point is related to the corresponding Euler class of the circle bundles over the space of polyhedral surfaces defined by the conical cotangent spaces at the ith vertex of the triangulation. While this is not an unexpected connection, the analogy with Witten–Kontsevich theory being obvious, we stress that the conical geometry adds to this property the possibility of a deep and explicit characterization of the Weil–Petersson form in terms of the edge lengths of the triangulation. This result is obtained by a subtle interplay between the geometry of polyhedral surfaces and 3D hyperbolic geometry, and it will be discussed in detail in Chap. 3, since it explicitly hints at the connection between polyhedral surfaces and quantum geometry in higher dimensions.

Chapter 3 deals with the interplay between polyhedral surfaces and 3D hyperbolic geometry mentioned above and the characterization of the Weil–Petersson form ω_{WP} on the space of polyhedral structures with given conical singularities. An important role in such a setting is played by the interesting recent results by G. Mondello regarding an explicit expression for the Weil–Petersson form for hyperbolic surfaces with geodesic boundaries. In order to construct a combinatorial representative of ω_{WP} for polyhedral surfaces, we exploit this result and the connection between similarity classes of Euclidean triangles and the triangulations of 3-manifolds by ideal tetrahedra. We describe this construction in detail, since it will also characterize a striking mapping between closed polyhedral surfaces and hyperbolic surfaces with geodesic boundaries. Such a mapping has a life of its own, strongly related to the geometry of the moduli space of pointed Riemann surfaces, and it provides a useful framework for discussing such matters as open/closed string dualities.

The content of Chap. 4 (Chap. 5 in this second edition) constitutes an introduction to the basic ideas of 2D quantum field theory and noncritical strings. This is classic material which nevertheless proves useful for illustrating the interplay between quantum field theory, the moduli space of Riemann surfaces, and the properties of polyhedral surfaces which are the *leitmotiv* of this LNP. At the root of this interplay lies 2D quantum gravity. It is well known that such a theory allows for two complementary descriptions: on the one hand, we have a conformal field theory (CFT) living on a 2D world-sheet, a description that emphasizes the geometrical aspects of the Riemann surface associated with the world-sheet, while on the other, the theory can be formulated as a statistical critical field theory over the space of polyhedral surfaces (dynamical triangulations). We show that many properties of such 2D quantum gravity models are related to a geometrical mechanism which allows one to describe a polyhedral surface with N_0 vertices as a Riemann surface with N_0 punctures dressed with a field whose charges describe discretized curvatures (related to the deficit angles of the triangulation). Such a picture calls into play the (compactified) moduli space of genus g Riemann surfaces with N_0 punctures $\mathfrak{M}_{g;N_0}$ and allows one to prove that the partition function of 2D quantum gravity is directly related to computation of the Weil–Petersson volume of $\mathfrak{M}_{g;N_0}$. By exploiting the large N_0 asymptotics of such Weil–Petersson volumes, recently characterized by Manin and Zograf, it is then easy to relate the anomalous scaling properties of pure 2D quantum gravity, the KPZ exponent, to the Weil–Petersson volume of $\mathfrak{M}_{g;N_0}$. This ultimately relates to the difficult problem of constructively characterizing the appropriate functional measures on spaces of Riemannian manifolds, often needed in the study of quantum gravity models and in the statistical mechanics of extended objects. We also address the more general case of the interaction of conformal matter with 2D quantum gravity and in particular the characterization of the associated KPZ exponents. By elaborating on the recent remarkable approach to the spectral theory over polyhedral surfaces due to A. Kokotov, we provide a general framework for analyzing KPZ exponents by discussing the scaling properties of the corresponding discretized Liouville theory.

In a rather general sense, polyhedral surfaces also provide a natural kinematical framework within which we can discuss open/closed string duality. A basic problem in such a setting is to provide an explanation of how open/closed duality is generated dynamically and in particular how a closed surface is related to a corresponding open surface, with gauge-decorated boundaries, in such a way that the quantization of this correspondence leads to an open/closed duality. Typically, the natural candidate for such a mapping is Strebel's theorem, which allows one to reconstruct a closed N-pointed Riemann surface M of genus g from the datum of the quadratic differential associated with a ribbon graph.

Are ribbon graphs, with the attendant BCFT techniques, the only key for addressing the combinatorial aspects of open/closed string duality? The results of Chap. 3 show that from a closed polyhedral surface we naturally get an open hyperbolic surface with geodesic boundaries. This gives a geometrical mechanism describing the transition between closed and open surfaces which, in a dynamical sense, is more interesting than Strebel's construction. Such a correspondence between closed polyhedral surfaces and open hyperbolic surfaces is indeed easily promoted to the corresponding moduli spaces: $\mathfrak{M}_{g;N_0} \times \mathbb{R}_+^N$, the moduli spaces of N_0-pointed closed Riemann surfaces of genus g whose marked points are decorated with the given set of conical angles, and $\mathfrak{M}_{g;N_0}(L) \times \mathbb{R}_+^{N_0}$, the moduli spaces of open Riemann surfaces of genus g with N_0 geodesic boundaries decorated by the corresponding lengths. Such a correspondence provides a nice kinematical setup for establishing an open/closed string duality, by exploiting the recent striking results by M. Mirzakhani on the relation between intersection theory over $\mathfrak{M}(g; N_0)$ and the geometry of hyperbolic surfaces with geodesic boundaries. The results in this chapter connect directly with many deep issues in 3D geometry, ultimately relating to the volume conjecture in hyperbolic geometry and to the role of knot invariants. This eventually brings us to the next topic we will discuss.

Indeed, Chaps. 5 and 6 (Chaps. 6 and 7 in this second edition) deal with the interplay between triangulated manifolds, knots, topological quantum field theory, and quantum computation. As Justin Roberts has aptly emphasized, the standard topological invariants were *created* in order to distinguish between things, and, owing to their intrinsic definitions, it is clear what kind of properties they reflect. For instance, the Euler number χ of a smooth, closed, and oriented surface $\mathscr{S}$ completely determines its topological type and can be defined as $\chi(\mathscr{S}) = 2 - 2g$, where g is the number of handles of $\mathscr{S}$.

On the other hand, quantum invariants of knots and 3-manifolds were *discovered*, but their indirect construction based on quantum group technology often hides information about the purely topological properties they are able to detect. What is lost at the topological level is, however, well paid back by the possibility of connecting this theory with a whole range of issues in pure mathematics and theoretical physics.

To the early connections such as quantum inverse scattering and exactly solvable models, it is worth adding the operator algebra approach used originally by Jones to define his knot polynomial. However, the most profitable development of the theory was the one suggested by Schwarz and formalized by Witten. Indeed, recognizing quantum invariants as partition functions and vacuum expectation values of physical observables in Chern–Simons–Witten topological quantum field theory provides a *physical* explanation for their existence and properties. Even more radically, one could speak of a conceptual explanation, as far as the topological origin of these invariants remains unknown. In this wider sense, quantum topology might be thought of as the mathematical substratum of an SU(2) CSW topological field theory, quantized according to the path integral prescription (the coupling constant $k \geq 1$ is constrained to be an integer related to the deformation parameter q by $q = \exp(\frac{2\pi i}{k+2})$).

The CSW environment provides the physical interpretation for quantum invariants, but in addition, it also includes all the historically distinct definitions. In particular, monodromy representations of the braid group appear in a variety of conformal field theories, since point-like "particles" confined in 2D regions evolve along braided worldlines. As a matter of fact, the natural extension of CSW theory to a 3-manifold $\mathscr{M}^3$ endowed with a nonempty 2D boundary $\partial\mathscr{M}^3$ induces on $\partial\mathscr{M}^3$ a specific quantized boundary conformal field theory, namely, the SU(2) Wess–Zumino–Witten (WZW) theory at level $\ell = k + 2$. The latter provides in turn the framework for dealing with $\mathrm{SU}(2)_q$-colored links presented as closures of oriented braids and associated with the Kaul unitary representation of the braid group. A further extension of this representation can be used to construct the quantum 3-manifold invariants explicitly within a purely algebraic setting.

Such quantities are essentially the Reshetikhin–Turaev–Witten invariants evaluated for 3-manifolds presented as complements of knots/links in the 3-sphere S^3, up to an overall normalization. Discretizations of manifolds appear here at a fundamental level, in particular, from $SU(2)$-decorated triangulations of 3D manifolds to triangulated boundary surfaces supporting a (boundary) conformal field theory.

Their use is relevant both in the characterization of the theory and in the actual possibility of computing the topological invariants under discussion. This computational role is a basic property, since the possibility of computing quantities of topological or geometric nature was recognized as a major achievement for quantum information theory by the Fields medalist Michael Freedman and coworkers.

Their *topological quantum computation* setting was designed to comply with the behavior of *modular functors* of 3D Chern–Simons–Witten (CSW) non-abelian topological quantum field theory (TQFT), the gauge group being typically SU(2). In physicists' language, such functors are partition functions and correlators of the quantum theory, and, owing to gauge invariance and invariance under diffeomorphisms, which freeze out local degrees of freedom, they share a global, topological character.

More precisely, the physical observables are associated with topological invariants of knots—the prototype of which is the Jones polynomial—and the generating functional is an invariant of the 3D ambient manifold, the Reshetikhin–Turaev–Witten invariant. We will discuss these matters in detail, with many illustrative examples and diagrams. We think that these case studies provide a good illustration of the richness of the subject, with a repertoire of mathematical techniques and physical concepts that may open up new and exciting territories for research.

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April 2011

Preface to the Second Edition

Ever since our LNP appeared 5 years ago, the field has undergone some important developments, among which we list the publication of the remarkable book by Bertrand Eynard *Counting Surfaces* (Progress in Mathematical Physics 70, Springer 2016), where one finds a deep framework for understanding the combinatorial threads among moduli space, matrix models, discretization of surfaces, and integrable systems. This time span has also seen progress in Liouville quantum gravity and in the analysis of the associated scaling laws, turning the elusive KPZ formula into a theorem (B. Duplantier, S. Sheffield, *Liouville Quantum Gravity and KPZ*, Inventiones Mathematicae 185(2), 333–393 (2011), and F. David, A. Kupiainen, R. Rhodes, V. Vargas, *Liouville Quantum Gravity on the Riemann Sphere* Commun. Math. Phys. (2016) 342: 869–907). Adding these topics to this second edition while appropriate in the spirit of the book was not possible for reason of space and the amount of background knowledge needed. We have chosen instead to make a single and focalized major addition to this new edition in the form of a rather vast chapter "The Quantum Geometry of Polyhedral Surfaces: Nonlinear σ Model and Ricci Flow." This includes sections on the geometry of nonlinear σ model, on the relation between Riemannian metric measure spaces and the dilaton, and on renormalization group and its connection with Ricci flow via the background field quantization method. The perturbative embedding of the Ricci flow in the renormalization group flow for NLσM is studied in considerable depth presenting methods and results not treated in any other book. We also discuss new techniques for studying this embedding. The reader can find most of the background knowledge needed carefully explained, with some additional material in an appendix on Riemannian geometry.

It is our pleasure and privilege to express our deep thanks to the many friends and colleagues who positively reacted to the first edition of this book.

Pavia, Italy — Mauro Carfora
Pavia, Italy — Annalisa Marzuoli
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Acknowledgments

We would like to thank Gaia for her patience with us over the years; our friends and colleagues Jan Ambjørn, Enzo Aquilanti, David Glickenstein, Giorgio Immirzi, Robert Littlejohn, Mario Rasetti, and Tullio Regge; and our young collaborators for so many discussions about the polyhedral side of mathematics and physics.

Tullio Regge would have been not surprised to learn that the subject of triangulations keeps on attracting with wonderful freshness the attention of mathematicians and physicists. The many facets of the theory, such as its beauty, sometimes its elegance, and certainly its unexpected connections with so many fields besides mathematics and physics, are a realm of geometry and the imagination that Tullio loved. This book is dedicated to his memory.

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