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Many mathematicians from all over the world have been involved in a way or another in C.I.M.E.'s activities over the years. The main purpose and mode of functioning of the Centre may be summarised as follows: every year, during the summer, sessions on different themes from pure and applied mathematics are offered by application to mathematicians from all countries. A Session is generally based on three or four main courses given by specialists of international renown, plus a certain number of seminars, and is held in an attractive rural location in Italy.

The aim of a C.I.M.E. session is to bring to the attention of younger researchers the origins, development, and perspectives of some very active branch of mathematical research. The topics of the courses are generally of international resonance. The full immersion atmosphere of the courses and the daily exchange among participants are thus an initiation to international collaboration in mathematical research.

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José Antonio Carrillo • Manuel del Pino • Alessio Figalli • Giuseppe Mingione • Juan Luis Vázquez

Nonlocal and Nonlinear Diffusions and Interactions: New Methods and Directions

Cetraro, Italy 2016

Matteo Bonforte • Gabriele Grillo
Editors

 Springer

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INTERNATIONAL MATHEMATICAL SUMMER CENTER

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Preface

We are honoured to have been the scientific organizers of the 2016 CIME Course “Nonlocal and nonlinear diffusions and interactions. New methods and directions”, which took place in Cetraro (Cosenza, Italy) on July 4–8, 2016, with the following sets of lectures:

- José Antonio Carrillo, Imperial College London, United Kingdom:
Swarming Models with Repulsive-Attractive Effects: Derivation, Model Hierarchies and Pattern Stability
- Manuel del Pino, Universidad de Chile, Chile:
Bubbling blow-up in critical parabolic problems
- Alessio Figalli, ETH Zürich, Switzerland:
Regularity results for local and nonlocal energy interactions
- Giuseppe Mingione, Università Parma, Italy:
Recent progresses in nonlinear potential theory
- Juan Luis Vázquez, Universidad Autónoma de Madrid, Spain:
Nonlinear Diffusion Equations with Fractional Laplacian Operators

During the school, there has been a special dinner to celebrate the 70th birthday of Juan Luis Vázquez. Juan Luis has always had a big involvement in Italian mathematics.

This CIME course has brought together some of the leading scholars in nonlinear partial differential equations (PDEs), with a special emphasis on rapidly developing topics which are of great and increasing interest both from the theoretical point of view and as concerns applications. This summer course has been a success even beyond our expectation. We are indebted to the distinguished speakers for giving such high-level lectures and also for the effort of writing this excellent quality set of lecture notes. We counted numerous attendance; there were more than 60 participants, mostly doctoral or postdoctoral students coming from more than 10 different countries. We would like also to thank them; without their active participation, this event would not have been so meaningful.

Throughout the whole week, the atmosphere has been especially nice, friendly and scientifically stimulating, mostly because of the active involvement of both the speakers and the participants.

We believe that the scientific level achieved in this course has been outstanding, but also quite adequate and accessible to the heterogeneous audience. We have received an extremely good feedback from the participants regarding both the practical organization and especially the high level of the courses and the very nice and stimulating atmosphere. It is worth mentioning that the speakers were not only amongst the top-level mathematicians in the respective fields, but also they were quite open to discussions with the students, friendly discussing with them after the lessons as well as in other moments.

We now briefly describe the single contributions included in this set of lecture notes.

José Antonio Carrillo contributed with a paper entitled “The geometry of diffusing and self-attracting particles in a one-dimensional fair-competition regime”. The authors consider aggregation-diffusion equations modelling particle interactions with nonlinear diffusion and nonlocal attraction, leading to variants of the Keller-Segel model of chemotaxis. Their analysis deals with the one-dimensional case, providing an almost complete classification. Amongst the topics dealt with, we stress the uniqueness of stationary states via suitable functional inequalities, asymptotic behaviour of solutions, convergence to equilibrium in Wasserstein distance in the critical singular kernel case and convergence to self-similarity for subcritical interaction strength. Interesting numerical simulations naturally complement the analytical part.

Manuel del Pino contributed with a paper entitled “Bubbling blow-up in critical parabolic problems”. The paper is devoted to the analysis of the blowup of solutions for some parabolic equations that involve *bubbling phenomena*. The term *bubbling* refers to the presence of families of solutions which at main order look like scalings of a single stationary solution which in the limit become singular but at the same time have an approximately constant energy level. This arise in various problems where critical loss of compactness for the underlying energy appears. Three main equations are studied, namely, the Sobolev critical semilinear heat equation in \mathbb{R}^n , the harmonic map flow from \mathbb{R}^2 into S^2 and the Patlak-Keller-Segel system in \mathbb{R}^2 .

Alessio Figalli contributed with the paper “Regularity theory for local and nonlocal minimal surfaces: an overview”. This review paper begins with a very useful general introduction to the classical theory of local minimal boundaries, explaining the main ideas behind the existence and regularity theory. This is extremely useful as it provides, in the second part of the note, a natural guideline in order to address the same kind of problems in the nonlocal context. Amongst the important results discussed in this note are (1) the existence of minimal surfaces, (2) the regularity theory of minimal graphs, (3) the regularity of flat minimal surfaces and (4) the analysis of blowups and minimal cones.

Giuseppe Mingione contributed with a paper entitled “Short Tales from Nonlinear Calderón-Zygmund Theory”. The paper is a very useful review of nonlinear Calderón-Zygmund theory, which aims at reproducing, in the nonlinear setting, the classical linear theory. This topic has large intersections with nonlinear potential

theory. A central theme of the paper relies on the idea that linear potential theory tools like Riesz potential can still be used to study fine properties of solutions to nonlinear equations.

Juan Luis Vázquez contributed with a paper entitled “The mathematical theories of diffusion. Nonlinear and fractional diffusion”. The paper begins with an historical survey on linear and nonlinear diffusions whose prototypes are the heat equation and the porous medium equation. Then the attention turns to new nonlinear and nonlocal diffusion models, coming from anomalous diffusions in physics that take into account long-range interactions and have also further applications in areas like biology, image processing or finance. The main points of the theory have been outlined: existence, uniqueness, regularity and asymptotic behaviour, focusing attention on the different techniques used in the proofs. The occurrence of stable diffusive patterns was stressed.

We had in Cetraro an interesting, rich and friendly atmosphere, created by the speakers, the participants and the CIME Organizers, in particular Elvira Mascolo (CIME Director) and Paolo Salani (CIME Secretary). We thank all of them warmly.

Madrid, Spain
Milano, Italy

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