Universitext

Universitext

Series editors

Sheldon Axler
San Francisco State University

Carles Casacuberta
Universitat de Barcelona

Angus MacIntyre
Queen Mary, University of London

Kenneth Ribet
University of California, Berkeley

Claude Sabbah
École polytechnique, CNRS, Université Paris-Saclay, Palaiseau

Endre Süli
University of Oxford

Wojbor A. Woyczyński
Case Western Reserve University

Universitext is a series of textbooks that presents material from a wide variety of mathematical disciplines at master's level and beyond. The books, often well class-tested by their author, may have an informal, personal even experimental approach to their subject matter. Some of the most successful and established books in the series have evolved through several editions, always following the evolution of teaching curricula, to very polished texts.

Thus as research topics trickle down into graduate-level teaching, first textbooks written for new, cutting-edge courses may make their way into *Universitext.*

More information about this series at http://www.springer.com/series/223

Norbert Steinmetz

Nevanlinna Theory, Normal Families, and Algebraic Differential Equations

Springer

Norbert Steinmetz
Fakultät für Mathematik
TU Dortmund
Dortmund
Germany

ISSN 0172-5939 ISSN 2191-6675 (electronic)
Universitext
ISBN 978-3-319-59799-7 ISBN 978-3-319-59800-0 (eBook)
DOI 10.1007/978-3-319-59800-0

Library of Congress Control Number: 2017942798

© Springer International Publishing AG 2017
This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.
The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.
The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Rolf Nevanlinna (1895–1980) Hans Wittich (1911–1984)

Photo: Privately owned

Introduction and Preface

Nevanlinna Theory, Normal Families, and Algebraic Differential Equations—how are these topics related to each other?

Zalcman's Re-scaling method set up a way to combine Nevanlinna Theory and Normal Families in both directions—to prove qualitative ('soft') results in Nevanlinna Theory by using Normal Family methods, and to prove normality criteria using results from Nevanlinna Theory as a pattern. Part of Schiff's *Normal Families* is dedicated to this interesting and fruitful connection. In the present Chap. 4, some old and new results in this direction are presented. Moreover, the connection of Normal Families with Algebraic Differential Equations is discussed on an elementary level.

In his seminal paper on the value distribution of meromorphic functions and their derivatives, Hayman opened a new field of application of Nevanlinna Theory. Some of his results on differential polynomials are outlined in his indispensable *Meromorphic functions*. He initiated a vast field of research in the 1970s and 80s, which to the author's knowledge has never appeared in book form. Nevanlinna himself was the first to apply his theory to problems of uniqueness of meromorphic functions, known as the Five- and Four-Value Theorems. The enormous progress initiated by Gundersen in the 1980s and early 90s has also never been presented in book form. In combination with introductory applications to Algebraic Differential Equations, these topics constitute Chap. 3.

The benefits of applying Nevanlinna Theory to the field of Algebraic Differential Equations were first recognised in the early 1930s, and then systematically since the 50s due to the pioneering work of Wittich. Apart from single chapters and a few remarks in the books [15, 84, 85, 96, 202], just Laine's monograph *Nevanlinna Theory and Complex Differential Equations* is dedicated to this field. Since the beginning of the new century much progress has been made in the context of Painlevé Differential Equations.[1] For the first time, Re-scaling and Normal Family

[1] We quote from the introduction to L.A. Rubel's *Entire and meromorphic functions* [146], where the author expressed the need to have more examples of interesting meromorphic functions. *One*

arguments were used to gain new insight into the nature of the so-called Painlevé transcendents and solutions to other algebraic differential equations. Inspired by Zalcman's Re-scaling Lemma and older ideas due to Yosida, the so-called Yosida classes entered the stage quite naturally. This material is presented in Chaps. 5 and 6, and in parts also in Chap. 4. The monograph [60] is concerned with Painlevé's Equations from the complex analytic point of view, but was published 'too early' to incorporate the newest developments, methods, and results. This, of course, may happen to every book. Regrettably, the theory of Differential Equations in the Complex Domain is not commonly accepted as a genuine part of Complex Analysis, though it is completely based on complex analytic techniques. Chapters 2, 3, and 4 urgently demonstrate that the theory of Algebraic Differential Equations provides indispensable tools, and their solutions often mark the range of validity of important results in Nevanlinna Theory.

Chapter 1 is included for the convenience of the reader. It provides material from classical Complex Analysis, which—apparently or seemingly—does not belong to the generally accepted background. This is particularly true for the topics of Ordinary Differential Equations and Asymptotic Expansions.

In Chap. 2 not only the classical Nevanlinna Theory is outlined, but also Cartan's Theory of Entire Curves and the Selberg–Valiron Theory of Algebroid Functions is briefly presented. This includes generalisations of the Second Main Theorem and various applications to problems in Complex Analysis which then become 'elementary',

The text is not written redundancy-free. Some of the problems are dealt with at an early stage using the methods at hand, and are picked up later on when new tools are available. Several examples and exercises require extensive computations, which in principle can be realised by hand. It is, however, much more convenient to use some computer algebra system like MAPLE. Non-experts like the author can use MAPLE like a separate sheet of paper to carry out auxiliary computations.

The present text was written within the first 2 years of my ultimate sabbatical, but has a long history. It developed from research in Nevanlinna Theory, Analytic Differential Equations, and related subfields of Complex Analysis starting in the late 1970s, with a break in the 90s, which were occupied by other activities. The choice of the material naturally depends on personal preferences, experiences, and skills. Rather than to aim at completeness of the presentation, I intended to explain the main ideas and results exemplarily.

I appreciate the support and criticism by friends and colleagues. Of course, the responsibility for mistakes and misinterpretations remains with me. I also profited very much from the survey by A. Eremenko and J. Langley within the English translation of the monograph *Value Distribution of Meromorphic Functions* by A.A. Gol'dberg and I.V. Ostrovskii, and, in particular, from helpful comments by the

promising source of such examples is the Painlevé transcendents. However, in spite of a growing literature on these functions, the unfortunate fact is that the "proofs" are incomplete and not rigorous [. . .]—and also in earlier and the earliest papers, one could add.

referees. It is thanks to them if the present version has become more reader-friendly than the version they commented on.

Finally, I gratefully acknowledge the seminal impact the wonderful books *Aufgaben und Lehrsätze aus der Analysis I und II* (Problems and Theorems in Analysis I and II) by G. Pólya and G. Szegö had on me since my time as a student. They accompanied me for life. I would be pleased if some of the exercises I posed in place of proofs and worked out examples came close to the spirit of Pólya and Szegö. Definitions and theorems form the skeleton of a theory, while it is brought to life by applications, examples, and exercises only.

Dortmund, Germany Norbert Steinmetz

Contents

Notation

$\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$	Set of positive, non-negative, all integers
$\mathbb{Q}$, $\mathbb{R}$	Set of rational, real numbers
$\mathbb{C}$, $\widehat{\mathbb{C}}$	Complex plane, Riemann sphere $\mathbb{C} \cup \{\infty\}$
Re, Im	Real- and imaginary part
$\mathbb{D}$, $\mathbb{H}$	Unit disc, upper or right half-plane
$\triangle_\delta(p)$	Local disc $\lvert z-p\rvert < \delta\lvert p\rvert^{-\beta}$ of 'radius' δ
$\mathfrak{P}$, $\mathfrak{Q}$	Set of poles and zeros (of the function in question)
$\mathfrak{C}$, $\mathscr{L}$	Cluster set, lattice
$\mathfrak{Y}^0_{\alpha,\beta}$, $\mathfrak{Y}_{\alpha,\beta}$	Yosida classes
$(\frac{s}{t}, \varpi)$-string	Sequence satisfying $p_{k+1} = p_k + (\varpi + o(1))p_k^{-\frac{s}{t}}$
$\lvert\cdot\rvert$, $\chi(\cdot,\cdot)$	Absolute value, chordal metric on $\widehat{\mathbb{C}}$
$\lVert\cdot\rVert$, $\lVert\cdot\rVert_\infty$	Euclidean, maximum norm on $\mathbb{R}^n$ and $\mathbb{C}^n$
dist (z, A)	Euclidean distance of z to the set A
$f^\sharp$, $f^{\sharp_\alpha}$	Spherical derivative, modified spherical derivative
S_f, $\{f, z\}$	Schwarzian derivative
$W(w_1, \ldots, w_n)$	Wronskian determinant of $w_1, \ldots, w_n$
$\wp$	Weierstraß P-function
(a, b, c, d)	Cross-ratio of $a, b, c, d \in \widehat{\mathbb{C}}$
$M(r,f)$, $m(r,f)$	Maximum modulus, Nevanlinna proximity function
$N(r,f)$, $n(r,f)$	Integrated, non-integrated counting function of poles
$T(r,f)$, $S(r,f)$	Nevanlinna characteristic, remainder term
$T_C(r,\mathfrak{g})$, $T_S(r,\mathfrak{f})$, $T_V(r,\mathfrak{f})$	Cartan, Selberg, Valiron characteristic
$\delta(a,f)$, $\vartheta(a,f)$	Deficiency, ramification index of a
$\varrho(f)$, $\varrho_a(f)$	Order of growth, exponent of convergence of a-points
ord	Order of Airy and Weber–Hermite solution
$\deg R$, $\deg_x R$	Degree, degree w.r.t. x of the rational function R
d_Ω, d_Ω	Degree, weight of the differential polynomial Ω
$\mathrm{res}_p f$	Residue of f at p
h_f	Phragmén–Lindelöf indicator function

$\asymp$	$\phi(r) \asymp \psi(r) \Leftrightarrow 1/C \leq \lvert\phi(r)/\psi(r)\rvert \leq C\ (r \geq r_0)$, also $a_n \asymp b_n \Leftrightarrow 1/C \leq \lvert a_n/b_n\rvert \leq C\ (n \geq n_0)$
$\sim$	$\phi(r) \sim \psi(r) \Leftrightarrow \phi(r)/\psi(r) \to 1$ as $r \to \infty$, also $f(z) \sim \sum_{k=0}^{\infty} c_k z^{-k}$, asymptotic series representing f
	End of proof, example, exercise, remark