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Metric Diffusion Along Foliations

Springer

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To Joanna, Jan, Julia & Zuzanna

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Introduction

In the last few years, the author of this book was looking for a kind of a deformation of a metric on a foliated Riemannian manifold, which will preserve, in some sense, the set of compact leaves. More precisely, given a foliation $\mathscr{F}$ on a compact Riemannian manifold (M, g), we are looking for a deformation D_t of the structure g or induced Riemannian metric d such that the limit of the set $\mathscr{C}$ of compact leaves under, for example, the Gromov–Hausdorff convergence is homeomorphic to $\mathscr{C}/\mathscr{F}$.

The first approach to this problem by warped foliations [22, 23] wasn't successful. Briefly speaking, for a compact foliated Riemannian manifold $(M, \mathscr{F}, g)$, and a smooth function $f : M \to (0, \infty)$ constant along the leaves of $\mathscr{F}$ the metric induced by the Riemannian structure g_f defined by

$$g_f(v, w) = f^2 g(v, w) \quad \text{for } v, w \text{ tangent to } \mathscr{F},$$

$$g_f(v, w) = g(v, w) \quad \text{if at least one of } v, w \text{ is perpendicular to } \mathscr{F}$$

is called the *warped metric*. f is called the *warping function*, while the metric space (M, d_f), where d_f is a metric induced by g_f, the *warped foliation*.

Let f_n be a sequence of warping functions converging uniformly to zero on a Reeb foliation $\mathscr{R}$ of an annulus $A = \{x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2\}$. In [23], the following was proved:

Theorem I.1 *The limit of warped Reeb foliation* $\lim^{\mathrm{GH}}_{n\to\infty}(M, d_{f_n})_{n\in\mathbb{N}}$ *under the Gromov–Hausdorff convergence is a singleton.*

The above example shows that the two boundary compact leaves, which are linked by a non-compact leaf, collapse, while warping, to the same point of the limit. The same can be observed for compact foliations. In [22] it was shown that the non-empty bad set of the compact foliation described by Epstein and Vogt in [10] collapses, in Gromov–Hausdorff topology, to the singleton.

The approach presented here uses more advanced tools. First, observe that the Wasserstein distance d_W (see [19]) of two Dirac measures on a Polish metric space (X, d) concentrated in $x, y \in X$ is equal to $d(x, y)$. Having d_W, with foliated heat diffusion operators D_t (introduced by L. Garnett in [11]) we define, for given time $t \geq 0$, the *metric* $\mathrm{D}_t d$ *diffused along a foliation* on a Riemannian manifold M equipped with a foliation $\mathscr{F}$ as the Wasserstein distance of Dirac measures diffused at time t. It occurs that $\mathrm{D}_t d$ defines the same topology on M as the initial one (Theorem 4.1).

In further considerations, we concentrate our attention on compact foliations. The reason is topological, namely, that the leaves have no ends, so that the nature of the heat kernel is well known. Notice that in the case of the manifolds with ends in general there is no knowledge on the heat kernel behaviour. On the other hand, the existence of a non-empty bad set can produce a number of problems on convergence of the family $(M, \mathrm{D}_t d)$.

The main purpose for this work is to try to answer the question, whether the family $(M, \mathrm{D}_t d)$ converges or not in d_{WH} to a closed subset of $\mathscr{P}(M)$. This will be the main subject of the studies presented here.

There is one more problem to settle. In the case of warped foliations, the Gromov–Hausdorff convergence was used. In the case of metric diffusion something else is more appropriate.

Let us denote by $\mathscr{P}(M)$ the set of all Borel probability measures on M. There is a natural isometric embedding of $(M, \mathrm{D}_t d)$ into $(\mathscr{P}(M), d_W)$ defined by

$$\iota_t : M \ni x \mapsto \mathrm{D}_t \delta_x \in \mathscr{P}(M).$$

Hence, for $t, s \geq 0$ we can consider $(M, \mathrm{D}_t d)$ and $(M, \mathrm{D}_s d)$ as the closed subsets of $\mathscr{P}(M)$ (which is compact if M is so), and we shall use the Hausdorff distance of $\iota_t(M)$ and $\iota_s(M)$ in $(\mathscr{P}(M), d_W)$. With the above in mind, we will write $(M, \mathrm{D}_t d)$ or simply M_t instead of $(\iota_t(M), d_W)$.

The book was planned to provide all necessary facts needed to understand the metric diffusion along compact foliations, that is some basic facts from the optimal transportation theory and the theory of foliations. Chapter 1 is devoted to the Wasserstein distance, Kantorovich Duality Theorem, and the metrization of the weak-* topology by the Wasserstein distance. Moreover, we prove some technical lemmas used in further considerations. In Chapter 2, we present some basics about foliations, holonomy, and heat diffusion. They are necessary to understand the notion of the metric diffusion. The compact foliations are discussed in Chapter 3 where we recall these facts which are essential for further considerations.

The main results are presented in Chapter 4. We define the metric diffusion $\mathrm{D}_t d$ and study the topology of the metric space $(M, \mathrm{D}_t d)$. The remaining pages are devoted to the limits of diffused metrics along compact foliations. We prove the necessary conditions for Wasserstein–Hausdorff convergence of the metric diffused along compact foliation with non-empty Epstein hierarchy. The first result (Theorem 4.6) provides an information about the geometry of the compact foliation, that is it describes the leaf volume growth near connected components of the bad sets. The second (Theorem 4.7) is rather measure-theoretic one. Enhancement of the necessary conditions presented in Theorem 4.6 and Theorem 4.7 allows us to formulate the sufficient condition of Wasserstein–Hausdorff convergence of metrics diffused along compact foliations of dimension one with finite Epstein hierarchy.

As a kind of supplement, we present some facts about the metric diffusion along non-compact foliations. We provide the full description of the limit for metrics diffused along foliation with at least one compact leaf on the two-dimensional torus T^2.

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