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Convergence and Summability of Fourier Transforms and Hardy Spaces

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To Márti
for her patience
and love

Preface

The main purpose of this book is to investigate the convergence and summability both of one-dimensional and multi-dimensional Fourier transforms.

It is known that the Fourier transform of $f \in L_1(\mathbb{R})$ is given by

$$\widehat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u) e^{-\imath x u} \, du \qquad (x \in \mathbb{R}),$$

where $\imath = \sqrt{-1}$. If $f \in L_p(\mathbb{R})$ for some $1 \leq p \leq 2$ and $\widehat{f} \in L_1(\mathbb{R})$, then the Fourier inversion formula holds:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(u) e^{\imath x u} \, du \qquad (x \in \mathbb{R}).$$

In other cases, we introduce the Dirichlet integrals $s_t f$ by:

$$s_T f(x) = \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \widehat{f}(u) e^{\imath x u} \, du.$$

One of the deepest results in harmonic analysis is Carleson's theorem [52, 187], i.e. for $f \in L_p(\mathbb{R})$, $1 < p < \infty$,

$$\lim_{T \to \infty} s_T f = f \qquad \text{a.e.}$$

The convergence holds also in the $L_p(\mathbb{R})$-norm. In this book, we do not prove Carleson's theorem as it is investigated exhaustively in several books (e.g. Arias de Reyna [8] or Grafakos [152] or Muscalu and Schlag [253]).

This convergence does not hold for $p = 1$. However, using a summability method, say the Fejér method, we can generalize these results. The most known result in summability theory is Lebesgue's theorem [212] about the Fejér means

[116], i.e. the Fejér means of an integrable function converge almost everywhere to the function:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T s_t f(x)\, dt = f(x) \qquad \text{a.e.}$$

The set of convergence was characterized as the Lebesgue points of f.

In this book, these results will be proved and generalized for one-dimensional and multi-dimensional Fourier transforms.

The book is structured as follows. At the beginning of each chapter a brief survey is given. A relatively new application of distribution theory is dealt with. More exactly, the theory of one- and multi-dimensional Hardy spaces is applied in Fourier analysis. In Chap. 1, one-dimensional Hardy spaces are discussed. Some inequalities for the Hardy-Littlewood maximal operator and the atomic decomposition of the Hardy spaces are verified. Then the interpolation spaces of the Hardy spaces are characterized. Using the atomic decomposition, we give a sufficient condition for an operator such that it is bounded from the Hardy space to L_p.

In Chap. 2, one-dimensional Fourier transforms are considered. Some basic facts about Fourier transforms and tempered distributions are given and the Fourier inversion formula is shown. We take a general summability method, the so-called θ-summation defined by a function $\theta : \mathbb{R} \to \mathbb{R}$. This summation contains all well-known summability methods, such as the Fejér, Riesz, Weierstrass, Abel, Picard, Bessel, Rogosinski, de La Vallée-Poussin summations. We prove that the maximal operators of the summability means are bounded from the Hardy space H_p to L_p, whenever $p > p_0$ for some $p_0 < 1$. The critical index p_0 depends on the summability method. For $p = 1$, we obtain a weak type inequality by interpolation which implies the almost everywhere convergence of the summability means. The one-dimensional version of the almost everywhere convergence and the weak type inequality are proved usually with the help of a Calderon-Zygmund type decomposition lemma. However, in higher dimensions, this lemma cannot be used for all cases investigated in this monograph. Our method, which can also be applied efficiently in higher dimensions, can be regarded as a new method to prove the almost everywhere convergence and weak type inequalities. The convergence theorem about Lebesgue points mentioned above will be proved as well. Finally, strong summability will be considered. Using the modern techniques of two- and multi-dimensional summability theorems, we give simple proofs for the strong summability results later in Chap. 5. After the classical books of Bary [16] and Zygmund [400], this is the first book which considers strong summability. Our method is very different from that of Zygmund and Bary. At the end of this chapter some summability methods are presented as special cases of the θ-summation.

In Chap. 3, different types of Hardy-Littlewood maximal operators and multi-dimensional Hardy spaces (denoted by $H_p^\square$ and H_p) are introduced. The methods of proofs for one and several dimensions are entirely different; in most cases, the theorems stated for several dimensions are much more difficult to verify. The atomic decomposition of each Hardy space and the interpolation spaces between these

Hardy spaces are verified. Sufficient conditions for an operator to be bounded from the Hardy space to L_p are given for each Hardy space. It is very interesting that in this result not only the one- and two-dimensional cases are different, but there is also an essential difference between the two- and multi-dimensional cases.

In Chap. 4, some simple facts about multi-dimensional Fourier transforms are mentioned and the norm and almost everywhere convergence of the multi-dimensional Dirichlet integrals are verified. In the next chapter, we will consider different summation methods for multi-dimensional trigonometric Fourier transforms. Basically, two types of summations will be introduced. In the first one, we take the integral in the summability means over the balls of ℓ_q and call it as ℓ_q-summability. In the literature, the cases $q = 1, 2, \infty$, i.e. the triangular, circular and cubic summability, are investigated. In the second version of summation, which is investigated in Chap. 6, we take the integrals over rectangles which is called rectangular summability. The θ-summability is considered in each version. In Chap. 5, it will be proved that the maximal operators of the ℓ_q-summability means are bounded from $H_p^{\square}$ to L_p, whenever $p > p_0$ for some $p_0 < 1$. Here the critical index p_0 depends on the summability method and the dimension. As we mentioned before, for $p = 1$, we obtain a weak type inequality by interpolation in this case, too, which implies again the almost everywhere convergence of the summability means. In two small sections, we will present some results about the circular Bochner-Riesz summability below the critical index, the proofs of which can be found in the books of Grafakos [152, 154, 155] and Lu and Yan [239]. Finally, new Lebesgue points are introduced and the convergence at these Lebesgue points is proved for functions from the Wiener amalgam spaces. One of the novelties of this book is that the Lebesgue points are studied also in the theory of multi-dimensional summability.

In the last chapter, rectangular θ-summability is investigated and similar results are proved as for the ℓ_q-summability. In this case, two types of convergence and maximal operators are considered, namely the restricted (convergence over the diagonal or more generally over a cone) and the unrestricted (convergence over $\mathbb{R}^d$). We show that the maximal operators of the rectangular summability means are bounded from H_p to L_p, whenever $p > p_0$ for some $p_0 < 1$. This implies the almost everywhere convergence of the summability means. The theorems about Lebesgue points are formulated in this case, too.

This book was aimed to be written so that it is as nearly self-contained as possible. However, it is assumed that the reader has some basic knowledge on analysis and functional analysis. Besides the classical results, recent results of the last 20–30 years are studied. I hope the book will be useful for researchers as well as for graduate or postgraduate students. Especially the first two chapters can be used well by graduate students and the other chapters rather by PhD students and researchers.

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Notations

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