

Applied and Numerical Harmonic Analysis

Series Editor

John J. Benedetto

University of Maryland
College Park, MD, USA

Editorial Advisory Board

Akram Aldroubi

Vanderbilt University
Nashville, TN, USA

Douglas Cochran

Arizona State University
Phoenix, AZ, USA

Hans G. Feichtinger

University of Vienna
Vienna, Austria

Christopher Heil

Georgia Institute of Technology
Atlanta, GA, USA

Stéphane Jaffard

University of Paris XII
Paris, France

Jelena Kovačević

Carnegie Mellon University
Pittsburgh, PA, USA

Gitta Kutyniok

Technische Universität Berlin
Berlin, Germany

Mauro Maggioni

Duke University
Durham, NC, USA

Zuowei Shen

National University of Singapore
Singapore, Singapore

Thomas Strohmer

University of California
Davis, CA, USA

Yang Wang

Michigan State University
East Lansing, MI, USA

More information about this series at <http://www.springer.com/series/4968>

Ferenc Weisz

Convergence and Summability of Fourier Transforms and Hardy Spaces

 Birkhäuser

Ferenc Weisz
Department of Numerical Analysis
Eötvös Loránd University
Budapest, Hungary

ISSN 2296-5009 ISSN 2296-5017 (electronic)
Applied and Numerical Harmonic Analysis
ISBN 978-3-319-56813-3 ISBN 978-3-319-56814-0 (eBook)
DOI 10.1007/978-3-319-56814-0

Library of Congress Control Number: 2017951129

Mathematics Subject Classification (2010): 42B08, 42A38, 42B30

© Springer International Publishing AG 2017

This book was advertised with a copyright holder in the name of the editor(s)/author(s) in error, whereas the publisher holds the copyright.

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Printed on acid-free paper

This book is published under the trade name Birkhäuser, www.birkhauser-science.com
The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

*To Márti
for her patience
and love*

Preface

The main purpose of this book is to investigate the convergence and summability both of one-dimensional and multi-dimensional Fourier transforms.

It is known that the Fourier transform of $f \in L_1(\mathbb{R})$ is given by

$$\widehat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u) e^{-ixu} du \quad (x \in \mathbb{R}),$$

where $\iota = \sqrt{-1}$. If $f \in L_p(\mathbb{R})$ for some $1 \leq p \leq 2$ and $\widehat{f} \in L_1(\mathbb{R})$, then the Fourier inversion formula holds:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(u) e^{ixu} du \quad (x \in \mathbb{R}).$$

In other cases, we introduce the Dirichlet integrals $s_T f$ by:

$$s_T f(x) = \frac{1}{\sqrt{2\pi}} \int_{-T}^T \widehat{f}(u) e^{ixu} du.$$

One of the deepest results in harmonic analysis is Carleson's theorem [52, 187], i.e. for $f \in L_p(\mathbb{R})$, $1 < p < \infty$,

$$\lim_{T \rightarrow \infty} s_T f = f \quad \text{a.e.}$$

The convergence holds also in the $L_p(\mathbb{R})$ -norm. In this book, we do not prove Carleson's theorem as it is investigated exhaustively in several books (e.g. Arias de Reyna [8] or Grafakos [152] or Muscalu and Schlag [253]).

This convergence does not hold for $p = 1$. However, using a summability method, say the Fejér method, we can generalize these results. The most known result in summability theory is Lebesgue's theorem [212] about the Fejér means

[116], i.e. the Fejér means of an integrable function converge almost everywhere to the function:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T s_t f(x) dt = f(x) \quad \text{a.e.}$$

The set of convergence was characterized as the Lebesgue points of f .

In this book, these results will be proved and generalized for one-dimensional and multi-dimensional Fourier transforms.

The book is structured as follows. At the beginning of each chapter a brief survey is given. A relatively new application of distribution theory is dealt with. More exactly, the theory of one- and multi-dimensional Hardy spaces is applied in Fourier analysis. In Chap. 1, one-dimensional Hardy spaces are discussed. Some inequalities for the Hardy-Littlewood maximal operator and the atomic decomposition of the Hardy spaces are verified. Then the interpolation spaces of the Hardy spaces are characterized. Using the atomic decomposition, we give a sufficient condition for an operator such that it is bounded from the Hardy space to L_p .

In Chap. 2, one-dimensional Fourier transforms are considered. Some basic facts about Fourier transforms and tempered distributions are given and the Fourier inversion formula is shown. We take a general summability method, the so-called θ -summation defined by a function $\theta : \mathbb{R} \rightarrow \mathbb{R}$. This summation contains all well-known summability methods, such as the Fejér, Riesz, Weierstrass, Abel, Picard, Bessel, Rogosinski, de La Vallée-Poussin summations. We prove that the maximal operators of the summability means are bounded from the Hardy space H_p to L_p , whenever $p > p_0$ for some $p_0 < 1$. The critical index p_0 depends on the summability method. For $p = 1$, we obtain a weak type inequality by interpolation which implies the almost everywhere convergence of the summability means. The one-dimensional version of the almost everywhere convergence and the weak type inequality are proved usually with the help of a Calderon-Zygmund type decomposition lemma. However, in higher dimensions, this lemma cannot be used for all cases investigated in this monograph. Our method, which can also be applied efficiently in higher dimensions, can be regarded as a new method to prove the almost everywhere convergence and weak type inequalities. The convergence theorem about Lebesgue points mentioned above will be proved as well. Finally, strong summability will be considered. Using the modern techniques of two- and multi-dimensional summability theorems, we give simple proofs for the strong summability results later in Chap. 5. After the classical books of Bary [16] and Zygmund [400], this is the first book which considers strong summability. Our method is very different from that of Zygmund and Bary. At the end of this chapter some summability methods are presented as special cases of the θ -summation.

In Chap. 3, different types of Hardy-Littlewood maximal operators and multi-dimensional Hardy spaces (denoted by H_p^\square and H_p) are introduced. The methods of proofs for one and several dimensions are entirely different; in most cases, the theorems stated for several dimensions are much more difficult to verify. The atomic decomposition of each Hardy space and the interpolation spaces between these

Hardy spaces are verified. Sufficient conditions for an operator to be bounded from the Hardy space to L_p are given for each Hardy space. It is very interesting that in this result not only the one- and two-dimensional cases are different, but there is also an essential difference between the two- and multi-dimensional cases.

In Chap. 4, some simple facts about multi-dimensional Fourier transforms are mentioned and the norm and almost everywhere convergence of the multi-dimensional Dirichlet integrals are verified. In the next chapter, we will consider different summation methods for multi-dimensional trigonometric Fourier transforms. Basically, two types of summations will be introduced. In the first one, we take the integral in the summability means over the balls of ℓ_q and call it as ℓ_q -summability. In the literature, the cases $q = 1, 2, \infty$, i.e. the triangular, circular and cubic summability, are investigated. In the second version of summation, which is investigated in Chap. 6, we take the integrals over rectangles which is called rectangular summability. The θ -summability is considered in each version. In Chap. 5, it will be proved that the maximal operators of the ℓ_q -summability means are bounded from H_p^\square to L_p , whenever $p > p_0$ for some $p_0 < 1$. Here the critical index p_0 depends on the summability method and the dimension. As we mentioned before, for $p = 1$, we obtain a weak type inequality by interpolation in this case, too, which implies again the almost everywhere convergence of the summability means. In two small sections, we will present some results about the circular Bochner-Riesz summability below the critical index, the proofs of which can be found in the books of Grafakos [152, 154, 155] and Lu and Yan [239]. Finally, new Lebesgue points are introduced and the convergence at these Lebesgue points is proved for functions from the Wiener amalgam spaces. One of the novelties of this book is that the Lebesgue points are studied also in the theory of multi-dimensional summability.

In the last chapter, rectangular θ -summability is investigated and similar results are proved as for the ℓ_q -summability. In this case, two types of convergence and maximal operators are considered, namely the restricted (convergence over the diagonal or more generally over a cone) and the unrestricted (convergence over \mathbb{R}^d). We show that the maximal operators of the rectangular summability means are bounded from H_p to L_p , whenever $p > p_0$ for some $p_0 < 1$. This implies the almost everywhere convergence of the summability means. The theorems about Lebesgue points are formulated in this case, too.

This book was aimed to be written so that it is as nearly self-contained as possible. However, it is assumed that the reader has some basic knowledge on analysis and functional analysis. Besides the classical results, recent results of the last 20–30 years are studied. I hope the book will be useful for researchers as well as for graduate or postgraduate students. Especially the first two chapters can be used well by graduate students and the other chapters rather by PhD students and researchers.

Acknowledgments

I am very grateful for the special atmosphere I feel amongst my colleagues at Eötvös Loránd University, Budapest. Their friendship and professional knowledge inspired me during my work a lot. My thanks are due to the Hungarian Scientific Research Funds (OTKA) No K115804 for supporting my research.

I would like to thank my colleague Péter Simon and my doctoral student, Kristóf Szarvas, for reading through the manuscript carefully and for their useful comments. I also thank Péter Kovács for his helpful assistance in creating the figures.

Above all, I am particularly indebted to my family, Márti, Ágoston, Gellért and Ambrus, the source of my happiness and inspiration. Their love and understanding are a continuous encouragement for me.

Notations

(i_n, j_n) , 221	$L_p(\mathbb{T})$, 4
1_H , 9	$L_p(\mathbb{X})$, 4
$B(c, h)$, 138	$L_p^r(\mathbb{R}^d)$, 393
$C(\mathbb{X})$, 4	$L_p^{loc}(\mathbb{R})$, 4
$C^\infty(\mathbb{R})$, 15	$L_{p,\infty}(\mathbb{R})$, 5
$C_0(\mathbb{R})$, 4	$M(p)$, 158
$C_c(\mathbb{R})$, 4	M_ω , 73, 83
$C_c^\infty(\mathbb{R})$, 15	$M_{\beta f}$, 12, 139
$C_u(\mathbb{R})$, 99	$M_{s f}$, 143
D_T , 91, 213	$M_{s,\beta f}$, 150
D_T^q , 213	Mf , 8
D_n , 86, 207	$N(p)$, 23, 153
D_n^q , 205	P , 22
D_s , 73, 83	$P^+ f$, 22
$D_T^1(i,j)$, 221	P^+ , 87
$E_q(\mathbb{R})$, 101	$P_{\square}^+ f$, 152
$E_q(\mathbb{R}^d)$, 318	P^- , 87
$E_q^r(\mathbb{R}^d)$, 393	P^d , 152
G_T , 219	$P^{\nabla} f$, 22
H^c , 32	$P_{\square}^{\nabla} f$, 152
$H_p^i(\mathbb{R}^d)$, 152	P_r , 22
$H_{p,\infty}(\mathbb{R})$, 22	P_r^d , 152
$H_{p,\infty}(\mathbb{R}^d)$, 152	$S(\mathbb{R})$, 16
$H_{p,\infty}^{\square}(\mathbb{R}^d)$, 152	$S(\mathbb{R}^d)$, 151
$H_p(\mathbb{R})$, 22	$S'(\mathbb{R})$, 18
$H_p(\mathbb{R}^d)$, 152	$S'(\mathbb{R}^d)$, 151
$H_p^{\square}(\mathbb{R}^d)$, 152	$S_{\square} f$, 153
$I(c, h)$, 9	$S_{\psi f}$, 153
K_T^θ , 96, 384	S_x , 159
$K_T^{q,\theta}$, 230	T_x , 73, 83
K_m , 23, 153	$U_{r,\beta}^{(1)} f$, 320
$L_0(\mathbb{R})$, 13	$U_{r,\beta}^{(2)} f$, 321
$L_p(\log L)^k(\mathbb{R}^d)$, 138	$U_{r,\beta} f$, 320

- $U_r^{(1)}f$, 320
 $U_f f$, 320
 $V_* f$, 183
 $W(C, \ell_q)(\mathbb{R})$, 98
 $W(C, \ell_q)(\mathbb{R}^d)$, 138
 $W(L_\infty, \ell_1)(\mathbb{R})$, 98
 $W(L_p(\log L)^k, \ell_q)(\mathbb{R}^d)$, 138
 $W(L_p, \ell_q)(\mathbb{R})$, 98
 $W(L_p, \ell_q)(\mathbb{R}^d)$, 138
 $W(L_p, c_0)(\mathbb{R})$, 98
 $W(L_p, c_0)(\mathbb{R}^d)$, 138
 $W(L_{p,\infty}, \ell_\infty)(\mathbb{R})$, 106
 $W(L_{p,\infty}, \ell_q)(\mathbb{R})$, 98
 $W(L_{p,\infty}, \ell_q)(\mathbb{R}^d)$, 138
 $W_I(L_p(\log L)^k, \ell_\infty)(\mathbb{R}^d)$, 143
 $W_I(L_p(\log L)^k, c_0)(\mathbb{R}^d)$, 143
 $W_I(L_p, \ell_\infty)(\mathbb{R}^d)$, 142
 $W_I(L_p, c_0)(\mathbb{R}^d)$, 143
 $[x_1, \dots, x_n]f$, 218
 $\dot{E}_q(\mathbb{R})$, 102
 $\dot{E}_q(\mathbb{R}^d)$, 318
 $\dot{E}_q^\tau(\mathbb{R}^d)$, 404
 $\dot{D}_p(I(0, 1))$, 116
 $\dot{D}_p(\mathbb{R})$, 107
 $\dot{D}_p^0(I(0, 1))$, 117
 ℓ_p , 138
 $\ell_p(\mathbb{Z})$, 4
 \hat{f} , 72, 77, 80, 81, 85, 203, 204
 \hat{u} , 83
 ι , 72
 λ , 4
 $\log^+ u$, 137
 \mathcal{F}^{-1} , 77
 $\mathcal{F}f$, 72, 203
 \mathcal{I} , 221
 $\mathcal{M}f$, 141, 142
 $\mathcal{M}(F)$, 157
 $\mathcal{M}_p^{(1)}f$, 142
 $\mathcal{M}^{(1)}f$, 140, 142
 $\mathcal{M}_p^{(2)}f$, 140
 $\mathcal{M}_1(F)$, 157
 $\mathcal{M}_i(F)$, 157
 $\mathcal{M}_p f$, 142
 $\mathcal{M}_p^{(1)}f$, 140
 $\mathcal{M}_p^{(2)}f$, 140
 $\mathcal{M}_{\tilde{p}} f$, 141
 $\mathcal{P}_{N(p)}$, 33
 $\phi^+ f$, 23, 153
 $\phi_{\square}^+ f$, 153
 $\phi_{\square}^\vee f$, 23, 153
 $\phi_{\square}^\nabla f$, 153
 $\sigma_*^\theta f$, 103, 398
 $\sigma_*^{q,\theta} f$, 264
 $\sigma_T^\theta f$, 96, 97, 384
 $\sigma_T^{q,\theta} f$, 230, 234
 $\sigma_{\square}^\theta f$, 386
 \sim , 23
 $\tau_*^\theta f$, 398
 $\tau_T^\theta f$, 398
 θ_0 , 226, 230
 $\theta_0^{(q)}$, 230
 \tilde{f} , 87
 $\Gamma(x)$, 153
 $\varrho(x, H)$, 32
 \vee , 120
 \wedge , 120
 \tilde{f} , 87
 $c_0(\mathbb{Z})$, 4
 $f * g$, 7
 f^{**} , 73
 f^\vee , 75, 203
 f_{\square} , 23, 153
 $f_{\tilde{m}}$, 152
 $f_{m,\square}$, 23, 153
 id , 220
 $m_s f$, 147
 $m_p f$, 105
 rI , 9
 $rI(x, h)$, 9
 $s_T^q f$, 213, 217
 $s_T f$, 91, 92, 213, 217
 $s_n f$, 86
 $s_n^q f$, 205
 $s_n f$, 207
 $u \cdot x$, 138
 u^\vee , 83
 \mathbb{C} , 4
 \mathbb{N} , 4
 \mathbb{P} , 4
 \mathbb{Q} , 4
 \mathbb{Q}_+ , 4
 \mathbb{R} , 4
 \mathbb{R}_+ , 4
 \mathbb{R}_+^d , 183
 \mathbb{R}_{ω}^d , 386
 \mathbb{T} , 4
 \mathbb{X} , 4
 \mathbb{Z} , 4
 soc , 120
 $(A_0, A_1)_{\theta,q}$, 49
 $H_{p,q}(\mathbb{R})$, 48
 J_k , 223
 $K(t, f, A_0, A_1)$, 48
 $L_{p,q}(\mathbb{R})$, 47

$N(p)$, 40, 156 $T_{*}f$, 13 $T_{\tilde{x}}$, 20 $V_{*}f$, 63 $\tilde{f}_{\tilde{x}}$, 20 $f_{\tilde{x}}$, 47 $s_{*}^q f$, 210, 217 \mathbb{R}_{ω}^d , 386

Contents

Part I One-Dimensional Hardy Spaces and Fourier Transforms

1 One-Dimensional Hardy Spaces	3
1.1 The L_p Spaces.....	4
1.2 Hardy-Littlewood Maximal Function.....	8
1.3 Schwartz Functions.....	15
1.4 Tempered Distributions and Hardy Spaces.....	18
1.5 Inequalities with Respect to Hardy Spaces.....	26
1.6 Atomic Decomposition.....	30
1.7 Interpolation Between Hardy Spaces.....	47
1.8 Bounded Operators on Hardy Spaces.....	60
2 One-Dimensional Fourier Transforms	71
2.1 Fourier Transforms.....	72
2.2 Tempered Distributions.....	82
2.3 Partial Sums of Fourier Series.....	85
2.4 Convergence of the Inverse Fourier Transform.....	90
2.5 Summability of One-Dimensional Fourier Transforms.....	95
2.6 Norm Convergence of the Summability Means.....	98
2.7 Almost Everywhere Convergence of the Summability Means.....	101
2.8 Boundedness of the Maximal Operator.....	108
2.9 Convergence at Lebesgue Points.....	112
2.10 Strong Summability.....	119
2.11 Some Summability Methods.....	130

Part II Multi-Dimensional Hardy Spaces and Fourier Transforms

3 Multi-Dimensional Hardy Spaces	137
3.1 Multi-Dimensional Maximal Functions.....	137
3.1.1 Hardy-Littlewood Maximal Functions.....	137
3.1.2 Strong Maximal Functions.....	142

3.2	Multi-Dimensional Tempered Distributions and Hardy Spaces	151
3.3	Inequalities with Respect to Multi-Dimensional Hardy Spaces	154
3.4	Atomic Decompositions	156
3.4.1	Atomic Decomposition of $H_p^\square(\mathbb{R}^d)$	156
3.4.2	Atomic Decomposition of $H_p(\mathbb{R}^d)$	157
3.5	Interpolation Between Multi-Dimensional Hardy Spaces	175
3.5.1	Interpolation Between the $H_p^\square(\mathbb{R}^d)$ Spaces	175
3.5.2	Interpolation Between the $H_p(\mathbb{R}^d)$ Spaces	176
3.6	Bounded Operators on Multi-Dimensional Hardy Spaces	183
3.6.1	Bounded Operators on $H_p^\square(\mathbb{R}^d)$	183
3.6.2	Bounded Operators on $H_p(\mathbb{R}^d)$	184
4	Multi-Dimensional Fourier Transforms	203
4.1	Fourier Transforms	203
4.2	Multi-Dimensional Partial Sums	204
4.3	Convergence of the Inverse Fourier Transform	213
4.4	Multi-Dimensional Dirichlet Kernels	218
4.4.1	Triangular Dirichlet Kernels	218
4.4.2	Circular Dirichlet Kernels	221
5	ℓ_q-Summability of Multi-Dimensional Fourier Transforms	229
5.1	The ℓ_q -Summability Means	229
5.2	Norm Convergence of the ℓ_q -Summability Means	234
5.2.1	Proof of Theorem 5.2.1 for $q = 1$ and $q = \infty$	235
5.2.2	Some Summability Methods	257
5.2.3	Further Results for the Bochner-Riesz Means	259
5.3	Almost Everywhere Convergence of the ℓ_q -Summability Means ...	264
5.3.1	Proof of Theorem 5.3.2	267
5.3.2	Proof of Theorem 5.3.3	309
5.3.3	Some Summability Methods	311
5.3.4	Further Results for the Bochner-Riesz Means	313
5.4	Convergence at Lebesgue Points	317
5.4.1	Circular Summability ($q = 2$)	317
5.4.2	Cubic and Triangular Summability ($q = \infty$ and $q = 1$)	319
5.5	Proofs of the One-Dimensional Strong Summability Results	374
6	Rectangular Summability of Multi-Dimensional Fourier Transforms	383
6.1	Norm Convergence of Rectangular Summability Means	383
6.2	Almost Everywhere Restricted Summability	386
6.3	Restricted Convergence at Lebesgue Points	393
6.4	Almost Everywhere Unrestricted Summability	398
6.5	Unrestricted Convergence at Lebesgue Points	404

Contents	xix
Bibliography	413
Applied and Numerical Harmonic Analysis (80 Volumes)	429
Index	433

List of Figures

Fig. 2.1	Dirichlet kernel D_T for $T = 5$	91
Fig. 2.2	Fejér kernel K_T^θ for $T = 5$	97
Fig. 4.1	Regions of the ℓ_q -partial sums for $d = 2$	205
Fig. 4.2	The Dirichlet kernel D_n^q with $d = 2, q = 1, n = 4$	206
Fig. 4.3	The Dirichlet kernel D_n^q with $d = 2, q = 2, n = 4$	206
Fig. 4.4	The Dirichlet kernel D_n^q with $d = 2, q = \infty, n = 4$	207
Fig. 4.5	The rectangular Dirichlet kernel with $d = 2, n_1 = 3, n_2 = 5$	208
Fig. 4.6	The projections P_1^+, P_2^+, Q_1 and Q	212
Fig. 4.7	The Dirichlet kernel D_T^q with $d = 2, q = 1, T = 4$	214
Fig. 4.8	The Dirichlet kernel D_T^q with $d = 2, q = 2, T = 4$	214
Fig. 4.9	The Dirichlet kernel D_T^q with $d = 2, q = \infty, T = 4$	215
Fig. 4.10	The rectangular Dirichlet kernel with $d = 2, T_1 = 3, T_2 = 5$	215
Fig. 5.1	The Fejér kernel $K_T^{q,\theta}$ with $d = 2, q = 1, T = 4$	231
Fig. 5.2	The Fejér kernel $K_T^{q,\theta}$ with $d = 2, q = \infty, T = 4$	231
Fig. 5.3	The Fejér kernel $K_T^{q,\theta}$ with $d = 2, q = 2, T = 4$	232
Fig. 5.4	The sets A_i	238
Fig. 5.5	Weierstrass summability function $\theta_0(t) = e^{-\ t\ _2^2/2}$	257
Fig. 5.6	Picard-Bessel summability function with $d = 2$	258
Fig. 5.7	Riesz summability function with $d = 2, \alpha = 1, \gamma = 2$	258
Fig. 5.8	The Bochner-Riesz kernel $K_T^{2,\alpha}$ with $d = 2, T = 4, \alpha = 1,$ $\gamma = 2$	260
Fig. 5.9	The Bochner-Riesz kernel $K_T^{2,\alpha}$ with $d = 2, T = 4,$ $\alpha = 1/10, \gamma = 2$	260
Fig. 5.10	Unboundedness of $\sigma_T^{2,\alpha}$	261
Fig. 5.11	Boundedness of $\sigma_T^{2,\alpha}$ when $d \geq 3$	261
Fig. 5.12	Boundedness of $\sigma_T^{2,\alpha}$ when $d = 2$	262
Fig. 5.13	Boundedness of $\sigma_T^{2,\alpha}$ when $d \geq 3$	262
Fig. 5.14	Open question of the boundedness of $\sigma_T^{2,\alpha}$ when $d \geq 3$	263

Fig. 5.15 Unboundedness of $\sigma_*^{2,\alpha}$ from $L_p(\mathbb{R}^d)$ to $L_{p,\infty}(\mathbb{R}^d)$ 313

Fig. 5.16 Boundedness of $\sigma_*^{2,\alpha}$ from $L_p(\mathbb{R}^2)$ to $L_{p,\infty}(\mathbb{R}^2)$ when $d = 2$ 314

Fig. 5.17 Boundedness of $\sigma_*^{2,\alpha}$ from $L_p(\mathbb{R}^d)$ to $L_{p,\infty}(\mathbb{R}^d)$ when $d \geq 3$ 315

Fig. 5.18 Open question of the boundedness of $\sigma_*^{2,\alpha}$ when $d \geq 3$ 315

Fig. 5.19 Almost everywhere convergence of $\sigma_T^{2,\alpha} f, f \in L_p(\mathbb{R}^d)$ 316

Fig. 5.20 The sets A_i 327

Fig. 6.1 The rectangular Fejér kernel K_T^θ with $d = 2, T_1 = 3, T_2 = 5$ 385

Fig. 6.2 The cone for $d = 2$ 387