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Mathematical Methods of Classical Physics

 Springer

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Symbols

A	Connection one-form, 69
$\mathcal{A}_G^{YM}(E)$	Subset of Yang–Mills G -connections, 69
$\text{Aut}_G(E)$	Gauge group of the vector bundle E with G -reduction $\mathcal{F}_G(E)$, 68
B	Magnetic field, 65
$\text{Diff}^+(M)$	Subgroup of orientation preserving diffeomorphisms of M , 53
$\text{Div } P$	Total divergence, 51
d^∇	Covariant exterior derivative, 65
$dvol$	Volume element on \mathfrak{S} , 48
E	(Total) energy, 6
E_a	Component of Euler–Lagrange operator, 50
E_{kin}	Kinetic energy, 5
E_{pot}	Potential energy, 6
E	Electric field, 65
$\mathcal{F}(E)$	Frame bundle of E , 64
F^∇	Curvature of a connection ∇ , 64
\tilde{f}	Legendre transform of a smooth function f , 28
g	Metric on M (Riemannian or pseudo-Riemannian), 6
H	Hamiltonian, 21
$\text{Isom}^+(N, h)$	Subgroup of orientation preserving isometries of the metric h on N , 63
J	(Noether) current, 60
J^0	Charge density, 60
$\text{Jet}^k(\mathfrak{S}, \mathcal{T})$	Jet bundle over \mathfrak{S} , 48
$j^k(f)$	Smooth section of the jet bundle $\text{Jet}^k(\mathfrak{S}, \mathcal{T})$, 48
J	Flux density, 60
\mathbb{K}	Field (in this book, $\mathbb{K} = \mathbb{R}$ or \mathbb{C}), 64
L	Length of the angular momentum vector \mathbf{L} , 14
\mathcal{L}	Lagrangian (function), 5

\mathcal{L}_2	Linearized Lagrangian, 31
\mathbf{L}	Angular momentum vector, 14
M	Smooth manifold (configuration space), 5
m	Mass of a point particle, 6
n	Dimension of M , 8
$\text{pr}^{(k)}Z$	k -th prolongation of Z , 54
\mathbf{p}	Momentum vector, 13
Q	(Noether) charge, 60
$(q^1, \dots, q^n, \hat{q}^1, \dots, \hat{q}^n)$	Induced/canonical local coordinates on TM associated with local coordinates (x^1, \dots, x^n) on M , 7
$(q^1, \dots, q^n, p_1, \dots, p_n)$	Darboux coordinates, 20
Ric	Ricci curvature tensor, 72
S	Action, 6
scal	Scalar curvature of (M, g) , 72
$S[f]$	Action functional of a classical field theory, 47
\mathfrak{S}	Source manifold (possibly with boundary), 45
t	Time, 5
\mathcal{T}	Target manifold, 47
TM	Tangent bundle of M , 5
T_v^μ	Component of the energy–momentum tensor, 76
V	Potential, 5
V_{eff}	Effective potential, 15
(x^1, \dots, x^n)	Local coordinates on an open subset $U \subset M$, 7
X_f	Hamiltonian vector field associated with a smooth function f , 20
$\mathfrak{X}(M)$	Set of all smooth vector fields on M , 11
X^{ver}	Vertical lift of X , 11
α	Euler–Lagrange one-form, 50
α_i	Component of the Euler–Lagrange one-form in some local coordinate system, 8
δ_{ij}	Kronecker delta. Its value is defined to be 1 if the indices are equal, and 0 otherwise, 33
γ	Smooth curve in M , 6
κ	Gravitational coupling constant, 72
λ	Liouville form, 20
Λ	Cosmological constant, 75
Ω	Hessian matrix of the potential V , 32
ω	Canonical symplectic form, 20
ω_0	Frequency of a small oscillation, 33
π	Canonical projection from TM to M , 7
$\tau(f)$	Tension of f , 62
τ_0	Period of a small oscillation, 33
ξ	Eigenvector of W , 35
$\langle \cdot, \cdot \rangle$	Euclidean scalar product on \mathbb{R}^n , 5

*	Hodge star operator, 65
Δ	Laplace operator, 63
$\dot{(\cdot)}$	Time derivate, $\dot{f}(t) = f'(t)$, $\ddot{f}(t) = f''(t)$, 2
∇	Covariant derivative or connection, 9
Σ	We use an adapted version of <i>Einstein's summation convention</i> throughout this book: Upper and lower indices appearing with the same symbol within a term are to be summed over. We write the symbol Σ to indicate whenever this convention is employed. Owing to the aforementioned convention, the summation indices can be (and are usually) omitted below the symbol Σ , 7
$(\cdot)^T$	Matrix transposition, 34