

Formal Matrices

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Formal Matrices

 Springer

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Symbols

E_{ij}	Matrix unit
$ A $	Determinant of a matrix A
$d(A)$	Determinant of a formal matrix A
$A \otimes B$	Kronecker product of matrices A and B
$M(n, R)$	Ring of all $n \times n$ matrices over the ring R
$M(n, R, \{s_{ijk}\})$ or $M(n, R, \Sigma)$	Ring of all formal matrices of order n over the ring R with multiplier system $\{s_{ijk}\} = \Sigma$
$M(n, R, s)$	Ring of all formal matrices of order n over the ring R with multiplier s
R°	Opposite ring to a ring R
$P(R)$	Prime radical of the ring R
$J(R)$	Jacobson radical of the ring R
$C(R)$	Center of the ring R
$Q(R)$	Maximal left ring of fractions of the ring R
$R \times S$	Direct product of rings R and S
$A_1 \oplus \dots \oplus A_n$	Direct sum of modules A_1, \dots, A_n
$\text{Ker } \varphi$ or $\text{Ker}(\varphi)$	Kernel of the homomorphism φ
$\text{Im } \varphi$ or $\text{Im}(\varphi)$	Image of the homomorphism φ
$\text{Rad } A$	Radical of the module A
$\text{Soc } A$	Socle of the module A
\bar{Z}	Closure of the submodule Z of the module A
\hat{A}	Injective hull of the module A
A^*	Character module of the module A
$\lim_{\rightarrow I} A_i$	Direct limit of modules A_i
$R\text{-Mod}$ ($\text{Mod-}R$)	Category of left (right) modules over the ring R
$P(R)$	Category of finitely generated projective R -modules
$P(A)$	Category of finitely A -projective modules
$\text{End}_R A$ or $\text{End}_R(A)$	Endomorphism ring of an R -module A
$\text{Biend}_R A$	Biendomorphism ring of an R -module A
$\text{End } G$	Endomorphism ring of an Abelian group G

$\text{Hom}_R(A, B)$	Homomorphism group from an R -module A into an R -module B
$\text{Hom}(G, H)$	Homomorphism group from an Abelian group G into an Abelian group H
$M \otimes_R N$	Tensor product of a right R -module M on a left R -module N
$K_0(R)$	Grothendieck group of the ring R
$K_0(A)$	Grothendieck group of the category of finitely A -projective modules
$K_1(R)$	Whitehead group of the ring R