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Stochastic Optimal Control in Infinite Dimension

Dynamic Programming and HJB Equations

With a Contribution by
Marco Fuhrman and Gianmario Tessitore

Springer

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To my parents and to Sara
– G.F.
To my parents and to my family
– F.G.
To my parents Franciszka Święch and Jerzy Święch
– A.Ś.

Preface

The main objective of this book is to give an overview of the theory of Hamilton–Jacobi–Bellman (HJB) partial differential equations (PDEs) in infinite-dimensional Hilbert spaces and its applications to stochastic optimal control of infinite-dimensional processes and related fields. Both areas have developed very rapidly in the last few decades. While there exist several excellent monographs on this subject in finite-dimensional spaces (see e.g., [263, 264, 385, 453, 468, 490, 576]), much less has been written in infinite-dimensional spaces. A good account of the infinite-dimensional case in the deterministic context can be found in [404] (see also [562] on optimal control of deterministic PDEs). Other books that touch on the subject are [29, 179, 468]. We attempt to fill this gap in the literature. Infinite-dimensional diffusion processes appear naturally and are used to model phenomena in physics, biology, chemistry, economics, mathematical finance, engineering, and many other areas (see e.g., [124, 177, 180, 372, 569]). This book investigates the PDE approach to their stochastic optimal control; however, infinite-dimensional PDEs can also be used to study other properties of such processes as large deviations, invariant measures, stochastic viability, stochastic differential games for infinite-dimensional diffusions, etc. (see [86, 177, 179, 249, 251, 261, 465, 467, 542, 544]).

To illustrate the main theme of the book, let us begin with a model distributed parameter stochastic optimal control problem. We want to control a process (called the state) given by an abstract stochastic differential equation in a real, separable Hilbert space H

$$\begin{cases} dX(s) = (AX(s) + b(s, X(s), a(s)))ds + \sigma(s, X(s), a(s))dW(s), & s > t \geq 0 \\ X(t) = x \in H, \end{cases}$$

where A is the generator of a C_0 semigroup in H, b, σ are some functions, and W is a so-called Q-Wiener process[1] in H. The functions $a(\cdot)$, called controls, are

[1] Q is a suitable self-adjoint positive operator in H, the covariance operator for W.

stochastic processes with values in some metric space Λ, which satisfy certain measurability properties. The above abstract stochastic differential equation is very general and includes various semilinear stochastic PDEs, as well as other equations which can be rewritten as stochastic functional evolution equations, for instance, stochastic differential delay equations. In a most typical optimal control problem we want to find a control $a(\cdot)$, called optimal, which minimizes a cost functional

$$J(t,x;a(\cdot)) = \mathbb{E}\left[\int_t^T l(s,X(s),a(s))ds + g(X(T))\right]$$

(for some $T > t$) among all admissible controls for some functions $l : [0,T] \times H \times \Lambda \to \mathbb{R}$, $g : H \to \mathbb{R}$.

The dynamic programming approach to the above problem is based on studying the properties of the so-called value function

$$V(t,x) = \inf_{a(\cdot)} J(t,x;a(\cdot))$$

and characterizing it as a solution of a fully nonlinear PDE, the associated HJB equation. Since the state $X(s)$ evolves in the infinite-dimensional space H, this PDE is defined in $[0,T] \times H$. The link between the value function V and the HJB equation is established by the Bellman principle of optimality known as the dynamic programming principle (DPP),

$$V(t,x) = \inf_{a(\cdot)} \mathbb{E}\left[\int_t^\eta l(s,X(s),a(s))ds + V(\eta,X(\eta))\right], \quad \text{for all } \eta \in [t,T].$$

Heuristically, the DPP can be used to define a two-parameter nonlinear evolution system and the associated HJB equation

$$\begin{cases} V_t + \langle Ax, DV\rangle + \inf_{a\in\Lambda}\{\frac{1}{2}\mathrm{Tr}\left[(\sigma(t,x,a)Q^{\frac{1}{2}})(\sigma(t,x,a)Q^{\frac{1}{2}})^* D^2V\right] \\ \qquad\qquad + \langle b(t,x,a), DV\rangle + l(t,x,a)\} = 0, \\ V(T,x) = g(x) \end{cases} \tag{1}$$

is its generating equation. Such a PDE is called infinite-dimensional or a PDE in infinitely many variables. We also call it unbounded since it has a term with an unbounded operator A which is well defined only on the domain of A. Other terms may also be undefined for some values of DV and D^2V, the Fréchet derivatives of V, which we may identify with elements of H and with bounded, self-adjoint operators in H respectively. In particular, the term $\mathrm{Tr}[(\sigma Q^{\frac{1}{2}})(\sigma Q^{\frac{1}{2}})^* D^2V]$ is well defined only if $(\sigma Q^{\frac{1}{2}})(\sigma Q^{\frac{1}{2}})^* D^2V$ is of trace class.

The main idea is to use the HJB equation to study the properties of the value function, find conditions for optimality, obtain formulas for synthesis of optimal feedback controls, etc. This approach turned out to be very successful for finite-dimensional problems because of its clarity and simplicity and thanks to the developments of the theory of fully nonlinear elliptic and parabolic PDEs, in particular the introduction of the notion of a viscosity solution and advances in regularity theory. However, even there many open questions remain, especially if the HJB equations are degenerate. We hope the dynamic programming approach will be equally valuable for infinite-dimensional problems even though a complete theory is not available yet.

Equation (1) is an example of a fully nonlinear second-order PDE of (degenerate) parabolic type. In this book, we will deal with more general and different versions of such equations and their degenerate elliptic counterparts. If Λ is a singleton, (1) is just a terminal value problem for a linear Kolmogorov equation. If Λ is not a singleton but the diffusion coefficient σ is independent of the control parameter a, (1) is semilinear. The theory of linear equations (and some special semilinear equations) has been studied by many authors and can be found in the books [29, 106, 179, 583]. The emphasis of this book is on semilinear and fully nonlinear equations.

There are several notions of solution applicable to PDEs in Hilbert spaces which are discussed in this book: classical solutions, strong solutions, mild solutions in the space of continuous functions, solutions in $L^2(\mu)$, and viscosity solutions. Classical solutions are the most regular ones. This notion of solution requires $C^{1,2}$ regularity in the Fréchet sense and imposes additional conditions so that all terms in the equation make sense pointwise for $(t,x) \in [0,T] \times H$. When classical solutions exist, we can apply the classical dynamic programming approach to obtain verification theorems and the synthesis of optimal feedback controls. Unfortunately, in almost all interesting cases it is not possible to find such solutions; however, they are very useful as a theoretical tool in the theory. The notions of strong solutions, mild solutions in the space of continuous functions, and solutions in $L^2(\mu)$ are introduced and studied only for semilinear equations and define solutions which have at least first derivative (in some suitable sense). Verification theorems and synthesis of optimal feedback controls can still be developed within their framework. The notion of viscosity solutions is the most general and applies to fully nonlinear equations; however, at the current stage there are no results on verification theorems and synthesis of optimal feedback controls.

Infinite-dimensional problems present unique challenges, and among them are the lack of local compactness and no equivalent of Lebesgue measure. This means that standard finite-dimensional elliptic and parabolic techniques which are based on measure theory cannot be carried over to the infinite-dimensional case. Moreover, the equations are mostly degenerate and contain unbounded terms which are singular. So the methods to find regular solutions to PDEs in infinite dimension like ours tend to be global and are based on semigroup theory, smoothing properties of transition semigroups (like the Ornstein–Uhlenbeck semigroups), fixed point techniques, and stochastic analysis. These methods are mostly restricted to equations of semilinear type. On the other hand, the notion of a viscosity solution is perfectly suited for fully nonlinear equations. It is local, and it does not require any regularity of solutions except continuity. As in finite dimension, it is based on a maximum principle through the idea of “differentiation by parts,” i.e., replacing the nonexisting derivatives of viscosity subsolutions (respectively, supersolutions) by the derivatives of smooth test functions at points where their graphs touch the graphs of subsolutions (respectively, supersolutions) from above (respectively, below). However, as the readers will see, this idea has to be carried out very carefully in infinite dimension.

This book includes chapters on the most important topics in HJB equations and the DPP approach to infinite-dimensional stochastic optimal control.

Chapter 1 contains the basic material on infinite-dimensional stochastic calculus which is needed in subsequent chapters. It is, however, not intended to be an introduction to stochastic calculus, which the reader is expected to have some familiarity with. Chapter 1 is included to make the book more self-contained. Most of the results presented there are well known; hence, we only provide references where the reader can find proofs and more information about concepts, examples, etc. We provide proofs only in cases where we could not find good references in the literature.

In Chap. 2, we introduce a general stochastic optimal control problem and prove a key result in the theory, namely the dynamic programming principle. We formulate it in an abstract and general form so that it can be used in many cases without the need to prove it again. Solutions of stochastic PDEs must be interpreted in various ways (strong, mild, variational, etc.), and our formulation of the DPP tries to capture this phenomenon. Our proof of the DPP is based on standard ideas; however, we have tried to avoid heavy probabilistic methods regarding weak uniqueness of solutions of stochastic differential equations. Our proof is thus more analytical.

We also introduce many examples of stochastic optimal control problems which can be studied in the framework of the approach presented in the book. They should give the readers an idea of the range and applicability of the material.

Chapter 3 is devoted to the theory of viscosity solutions. The reader should keep in mind the following principle when it comes to unbounded PDEs in infinite dimension: There is no single definition of viscosity solutions that applies to all equations. This is due to the fact that there are many different PDEs which contain different unbounded operators and terms which are continuous in various norms. Also the solutions have to be continuous with respect to weaker topologies. However, the main idea of the notion of viscosity solutions is always the same as we described before. What changes is the choice of test functions, spaces, topologies, and the interpretation of various terms in the equation. In this book, we focus on the notion of a so-called B-continuous viscosity solution which was introduced by Crandall and Lions in [141, 142] for first-order equations and later adapted to second-order equations in [539]. The key result in the theory is the comparison principle, which is very technical. Its main component is the so-called maximum principle for semicontinuous functions. The proof of such a result in finite dimension was first obtained in [370] and was later simplified and generalized in [137–139, 360]. It is heavily based on measure theory and is not applicable to infinite dimension. Thus, the theory uses a finite-dimensional reduction technique introduced by Lions in [413]. It restricts the class of equations which can be considered; in particular, they have to be highly degenerated in the second-order terms. We present three techniques to obtain the existence of viscosity solutions. The first and most important for this book is the DPP and the stochastic optimal control interpretation, showing directly that the value function is a viscosity solution. This technique applies to HJB equations. The other techniques are finite-dimensional approximations and Perron's method. Both can be applied to more general equations, for instance, Isaacs equations associated to two-player, zero-sum stochastic differential games; however, they have limitations of their own. Moreover, we discuss other topics in the theory of viscosity solutions such as consistency and singular perturbations. Several special equations are also studied in this book because of their importance and because they are good examples to show how the definition of viscosity solutions and some techniques can be adjusted to particular cases. They are the HJB equations for the optimal control of the Duncan–Mortensen–Zakai equation, stochastic Navier–Stokes equations, and stochastic boundary control. In particular, the last one also contains ideas on how to handle HJB equations which may be nondegenerate, for instance, if Q is not of trace class. Finally, we present applications to the infinite-dimensional Black–Scholes–Barenblatt equations of mathematical finance.

Chapter 4 is devoted to the theory of mild and strong solutions in spaces of continuous functions through fixed point techniques based on the smoothing properties of transition semigroups such as Ornstein–Uhlenbeck semigroups. This theory applies only to semilinear equations, i.e., when the coefficient σ does not depend on the control parameter a, and historically it was the first approach introduced in the literature. The theory was initiated by Barbu and Da Prato [29] and later improved and developed in various papers, see e.g., [89, 90, 105, 107, 302, 307, 308, 311].

Chapter 4 is divided into four main parts. In the first part (Sects. 4.2 and 4.3), we present the basic tools needed for the analysis: the theory of generalized gradients and the smoothing of transition semigroups. In the second part (Sects. 4.4–4.7), we develop the theory for a general type of semilinear HJB equation (parabolic and elliptic) without connection with optimal control problems. The main idea behind this approach is the following. Consider the HJB equation (1) in the semilinear case when the coefficient σ is time-independent:

$$\begin{cases} V_t + \mathcal{A}V + \inf_{a\in\Lambda}\{\langle b(t,x,a), DV\rangle + l(t,x,a)\} = 0, \\ V(T,x) = g(x), \end{cases} \tag{2}$$

where $\mathcal{A}$ is the linear operator

$$\mathcal{A}\varphi = \langle Ax, D\varphi\rangle + \frac{1}{2}\mathrm{Tr}\left[(\sigma(x)Q^{\frac{1}{2}})(\sigma(x)Q^{\frac{1}{2}})^{*}D^2\varphi\right].$$

If such an operator generates a semigroup $e^{t\mathcal{A}}$ then, by the variation of constants formula, one can rewrite Eq. (2) in the integral form as

$$V(t,x) = e^{(T-t)\mathcal{A}}g(x) + \int_t^T \left(e^{(T-s)\mathcal{A}}F(s,\cdot)\right)(x)ds,$$

where $F(s,x) := \inf_{a\in\Lambda}\{\langle b(s,x,a), DV\rangle + l(s,x,a)\}$. The solution of this integral equation is called a mild solution and is obtained by fixed point techniques. To define it, the solution must at least have a first-order spatial Gâteaux derivative, possibly only in some directions needed to give sense to the nonlinear term, the so-called G-derivative. Thus, one needs suitable smoothing properties of the semigroup $e^{t\mathcal{A}}$ (which is the Ornstein–Uhlenbeck semigroup in the simplest case). Since this semigroup is not strongly continuous, except in very special cases, one needs to use the theory of π-semigroups introduced in [493] or that of weakly continuous (or $\mathcal{K}$-continuous) semigroups [101, 108, 301]. Sects. 4.4 and 4.5 consider a general type of operator $\mathcal{A}$, possibly depending on t, while Sects. 4.6 and 4.7 focus on the case when $\mathcal{A}$ is of Ornstein–Uhlenbeck type, where stronger results can be proved.

In the third part (Sect. 4.8), we develop a connection with stochastic optimal control problems. The fact that mild solutions have a first-order spatial derivative allows us to give a meaning to formulae for optimal feedbacks. However, the proofs of the verification theorems and optimal feedback formulae cannot be done straightforwardly as one needs to apply Itô's formula in infinite dimension, which requires smooth functions. For this reason (following [307]), we introduce the notion of a strong solution of the HJB equation (2) as a suitable limit of classical solutions and prove that any mild solution is also a strong solution.

The fourth and last part of the chapter (Sects. 4.9 and 4.10) deals with some special equations. In Sect. 4.9, we show how the techniques developed in the previous sections can be adapted to HJB equations and analysis of optimal control problems for the stochastic Burgers equation, stochastic Navier–Stokes equations and stochastic reaction diffusion equations. In Sect. 4.10, we discuss some equations for which explicit representations of the solutions can be found. Such cases are always of interest in applications.

Chapter 5 is devoted to a relatively new and promising theory of mild and strong solutions in spaces of L^2 functions with respect to a suitable measure μ (see [3, 4, 125, 299]). The contents of this chapter are similar to the previous one as the main ideas behind the definition of mild and strong solutions of HJB equations are the same. The difference is in the fact that the reference space is not the space of continuous functions but the space of square-integrable functions with respect to the measure μ. The results are similar: existence and uniqueness of solutions of HJB equations through fixed point arguments, verification theorem through approximations, and existence of optimal feedbacks. The advantage of this approach is that the results require weaker assumptions on the data, thus enlarging the range of possible applications, including the control of delay equations; however, at a cost of weaker statements, for example, the first-order spatial derivative is now defined in a Sobolev weak sense and is not in general a Gâteaux or Fréchet derivative. The main tools used here are the theory of invariant measures for infinite-dimensional stochastic differential equations and the properties of transition semigroups in the space of integrable functions with respect to such measures.

Chapter 6 is devoted to a different and in many respects complementary technique of Backward Stochastic Differential Equations (BSDEs). The chapter was written independently and autonomously by M. Fuhrman and G. Tessitore, who are well-recognized experts in the field. We are grateful for their invaluable contribution. BSDEs are Itô type equations in which the initial condition is replaced by a final condition and a new unknown process appears corresponding to a suitable martingale term. In the nonlinear, finite-dimensional case BSDEs were introduced in [476] while their direct connection with optimal stochastic control was first investigated in [212] and [483]. Since then, the general theory of BSDEs has developed considerably, see [78, 80, 210, 378, 421, 475]. Besides stochastic control, applications were given to many fields, for instance, to optimal stopping, stochastic differential games, nonlinear partial differential equations and many topics related to mathematical finance. Infinite-dimensional BSDEs have also been considered, see for instance, [130, 285, 331, 351, 477]. The interest for us is that BSDEs provide an alternative way to represent the value function of an optimal control problem and consequently to study the corresponding HJB equation and to solve the control problem. It turns out that the most suitable notion of solution for the HJB equation is, in this context, that of a mild solution on spaces of continuous functions but, unlike in Chap. 4, the BSDE method seems particularly adapted to treating degenerate cases in which the transition semigroup has no smoothing properties. The price to pay is that normally we need more regular coefficients and a structural condition (imposing, roughly speaking, that the control acts within the image of the noise). If these requirements are satisfied, the BSDE techniques are revealed to be very flexible. In particular, in Chap. 6 we will show how they allow us to treat both parabolic and elliptic HJB equations (see [77, 286, 352, 436, 478]). The parabolic case is treated for nonconstant diffusion and Lipschitz nonlinearity, while the elliptic case is considered for a constant diffusion operator with locally Lipschitz (with respect to the gradient) nonlinearity and a mild dissipativity assumption (with respect to the solution). We also report (without proofs) the results of [286] concerning elliptic HJB equations with nonconstant diffusion, a globally Lipschitz Hamiltonian and strong dissipativity. A detailed discussion of the literature on BSDEs in infinite dimension is contained in the bibliographical notes of Chap. 6.

It is impossible to cover all aspects of the theory of HJB equations in infinite dimension and its connections to stochastic optimal control. In particular, the theory of integro-PDEs is an emerging area which is not presented in the book. We do not discuss first-order equations and extensions to Banach spaces. Equations in the space of probability measures is another emerging topic. We have chosen a selection of topics which give a broad overview of the field and enough information so that the readers can start exploring the subject on their own. There are already enough important applications to justify the interest in the subject. The readers should not be restricted to the boundaries drawn by the book. We hope that this book will spur interest and research in the field among theoretical and applied mathematicians, and that it will be useful to all kinds of scientists and researchers working in areas related to stochastic control.

Suggestions for reading. The readers who are familiar with probability and stochastic analysis in infinite dimension can skip Chap. 1 and go directly to Chap. 2. Chapter 2 is needed for the understanding of the other chapters; however, some material in Sect. 2.3 related to technical details of the proof of the dynamic programming principle can be omitted during the first reading. Chaps. 3–6 are to a large extent independent of each other, and hence the reader can pass from Chap. 2 directly to any of them.

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